

DAA/Langley

DEPARTMENT OF MECHANICAL ENGINEERING AND MECHANICS
SCHOOL OF ENGINEERING
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23508

RADIATIVE INTERACTIONS IN TRANSIENT ENERGY
TRANSFER IN GASEOUS SYSTEMS

By

Surendra N. Tiwari, Principal Investigator

(NASA-CR-176644) RADIATIVE INTERACTIONS IN
TRANSIENT ENERGY TRANSFER IN GASEOUS SYSTEMS
Progress Report (Old Dominion Univ.) 103 p
HC A06/MF A01

N86-21798

CSCL 20D

Unclass

G3/34 16571

Progress Report
For the period ending November 31, 1985

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23665

Under
Research Grant NAG-1-423
A. Kumar and J. P. Drummond, Technical Monitors
HSAD-Computational Methods Branch

December 1985



DEPARTMENT OF MECHANICAL ENGINEERING AND MECHANICS
SCHOOL OF ENGINEERING
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23508

RADIATIVE INTERACTIONS IN TRANSIENT ENERGY
TRANSFER IN GASEOUS SYSTEMS

By

Surendra N. Tiwari, Principal Investigator

Progress Report
For the period ending November 31, 1985

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23665

Under
Research Grant NAG-1-423
A. Kumar and J. P. Drummond, Technical Monitors
HSAD-Computational Methods Branch

Submitted by the
Old Dominion University Research Foundation
P.O. Box 636
Norfolk, VA 23508

December 1985

PREFACE

This report covers the work completed on the research project "Analysis and Computation of Internal Flow Field in a Scramjet Engine," for the period ending November 30, 1985. The work was supported by the NASA Langley Research Center (Computational Methods Branch of the High-Speed Aerodynamics Division) through research grant NAG-1-423. The grant was monitored by Dr. A. Kumar and Mr. J. P. Drummond of the High-Speed Aerodynamics Division.

ABSTRACT

Analyses and numerical procedures are presented to investigate the radiative interactions in transient energy transfer processes in gaseous systems. The nongray radiative formulations are based on the wide-band model correlations for molecular absorption. Various relations for the radiative flux are developed; these are useful for different flow conditions and physical problems. Specific plans for obtaining extensive results for different cases are presented. The methods presented in this study can be extended easily to investigate the radiative interactions in realistic flows of hydrogen-air species in the scramjet engine.

TABLE OF CONTENTS

	<u>Page</u>
PREFACE.....	ii
ABSTRACT.....	iii
LIST OF SYMBOLS.....	v
1. INTRODUCTION.....	1
2. BAND ABSORPTION AND CORRELATIONS.....	2
3. RADIATIVE FLUX EQUATIONS.....	9
4. BASIC FORMULATION FOR TRANSIENT PROCESSES.....	18
5. A SPECIAL CASE OF TRANSIENT INTERACTION.....	21
6. METHOD OF SOLUTIONS.....	25
6.1 Optically Thin Limit.....	25
6.2 Large Path Length Limit.....	27
6.3 Numerical Solutions of Governing Equations.....	29
7. RADIATIVE INTERACTION IN LAMINAR FLOW.....	35
7.1 Basic Formulation.....	35
7.2 Optically Thin Limit.....	42
7.3 Large Length Limit.....	47
8. PLANS FOR SPECIFIC RESULTS.....	57
9. CONCLUDING REMARKS.....	59
REFERENCES.....	60
APPENDIX A: EXPONENTIAL INTEGRALS AND EXPRESSIONS FOR RADIATIVE FLUX.....	64
APPENDIX B: ALTERNATE FORMS OF RADIATIVE FLUX EQUATIONS.....	66
APPENDIX C: DEFINITION AND EVALUATION OF INTEGRAL FUNCTIONS..	72
APPENDIX D: INFORMATION FOR NUMERICAL PROCEDURE.....	79
APPENDIX E: EVALUATION OF CONSTANTS FOR STEADY LAMINAR FLOWS.....	82
APPENDIX F: INTEGRAL FUNCTIONS FOR TRANSIENT LAMINAR FLOWS.....	91

TABLE OF CONTENTS - Concluded

LIST OF FIGURES

<u>Figure</u>		<u>Page</u>
3.1	Plane radiating layer between parallel boundaries.....	9a
4.1	Physical model and coordinate system.....	19a
7.1	Physical model and coordinate system for flow of radiating gases between parallel plates.....	35a

LIST OF SYMBOLS

A	band absorptance , cm^{-1}
$A(u,\beta)$	band absorptance of a wide band, cm^{-1}
$A_n(u,\beta)$	band absorptance of a narrow band, cm^{-1}
A_0	band width parameter, cm^{-1}
C_0	correlation parameter, $\text{atm}^{-1} - \text{cm}^{-1}$
e_ω	Planck's function, $(\text{W cm}^{-2})/\text{cm}^{-1}$
$e_{\omega_c}, e_{\omega_0}$	Planck's function evaluated at the wave ω_c, ω_0
$E_n(t)$	exponential integral of order n
H	gas property for the large path length limit
L	distance between plates
M	radiation-conduction interaction parameter for the large path length limit
N	optically thin radiation-conduction interaction parameter
P	pressure, atm
P_i	partial pressure of the i th species
q_r	total radiative flux, W/cm^2
$q_{r\omega}$	spectral radiative flux, $(\text{W}/\text{cm}^2)/\text{cm}^{-1}$
Q	nondimensional radiative heat flux, Eq. (4.4)
S	integrated intensity of a wide band, $\text{atm}^{-1}\text{-cm}^{-2}$
S_j	intensity of the j th spectral line
t	line structure parameter = $\beta/2 = \pi \gamma_L/d$
T	equilibrium temperature, K
T_1, T_2	wall temperatures, K
u	dimensionless coordinate = SX/A_0 or SPy/A_0
u_0	dimensionless path length = SPL/A_0
X	pressure path length = Py

y	transverse coordinate, cm
β	line structure parameter
γ_j	line half width, cm^{-1}
θ	dimensionless temperature, Eq. (5.4)
K_ω	equilibrium spectral absorption coefficient, cm^{-1}
τ	optical coordinate
τ_0	optical thickness
ξ	dimensionless coordinate = $y/L = u/u_0$
ω	wave number, cm^{-1}
ω_0, ω_c	wave number at the band center, cm^{-1}

1. INTRODUCTION

In the past two decades, a tremendous progress has been made in the field of radiative energy transfer in nonhomogeneous nongray gaseous systems. As a result, several useful books [1-18] and review articles [19-26] have become available for engineering, meteorological, and astrophysical applications. In the sixties and early seventies, radiative transfer analyses were limited to one-dimensional cases. Multidimensional analyses and sophisticated numerical procedures emerged in the mid-to-late seventies. Today, the field of radiative energy transfer in gaseous systems is getting an ever increasing attention because of its applications in the areas of the earth's radiation budget studies and climate modeling, fire and combustion research, entry and reentry phenomena, hypersonic propulsion and defense-oriented research.

In most studies involving combined mass, momentum, and energy transfer, the radiative transfer formulation has been coupled only with the steady processes. The goal of this research is to include the nongray radiative formulation in the general unsteady governing equations and provide the step-by-step analysis and solution procedure for several realistic problems. The specific objective of the present study is to investigate the one-dimensional transient radiative transfer in a nongray gaseous system. In the future work, the present analysis will be extended (in a systematic manner) to the problems of combined transfer processes in chemically reacting flows.

For the present study, the information on band absorption and correlation is summarized in section 2 and fundamental radiative flux equations are presented in section 3. The basic formulation for the transient radiation is given in section 4, and this is applied to a special case in section 5. The

solution procedures are described in section 6, and plans for obtaining specific results are presented in section 7.

2. BAND ABSORPTION AND CORRELATIONS

The study of radiative transmission in nonhomogeneous gaseous systems requires a detailed knowledge of the absorption, emission, and scattering characteristics of the specific species under investigation. In absorbing and emitting mediums, an accurate model for the spectral absorption coefficient is of vital importance in the correct formulation of the radiative flux equations. A systematic representation of the absorption by a gas, in the infrared, requires the identification of the major infrared bands and evaluation of the line parameters (line intensity, line half-width, and spacing between the lines) of these bands. The line parameters depend upon the temperature, pressure and concentration of the absorbing molecules and, in general, these quantities vary continuously along a nonhomogeneous path in the medium. In recent years, considerable efforts have been expended in obtaining the line parameters and absorption coefficients of important atomic and molecular species [27-30].

For an accurate evaluation of the transmittance (or absorptance) of a molecular band, a convenient line model is used to represent the variation of the spectral absorption coefficient. The line models usually employed are Lorentz, Doppler, and Voigt line profiles. A complete formulation (and comparison) of the transmittance and absorptance by these line profiles is given in [22-26]. In a particular band consisting of many lines, the absorption coefficient varies very rapidly with the frequency. Thus, it becomes very difficult and time-consuming task to evaluate the total band absorptance over the actual band contour by employing an appropriate line

profile model. Consequently, several approximate band models (narrow as well as wide) have been proposed which represent absorption from an actual band with reasonable accuracy [22-26, 31-40]. Several continuous correlations for the total band absorption are available in literature [22-26, 36-40]. These have been employed in many nongray radiative transfer analyses with varying degree of success [22-26, 41]. A brief discussion is presented here on the total band absorption, band models, and band absorptance correlations.

The absorption within a narrow spectral interval of a vibration rotation band can quite accurately be represented by the so-called "narrow band models." For a homogeneous path, the total absorptance of a narrow band is given by

$$A_N = \int_{\Delta\omega} [1 - \exp(-k_\omega X)] d\omega \quad (2.1)$$

where k_ω is the volumetric absorption coefficient, ω is the wave number, and $X = py$ is the pressure path length. The limits of integration in Eq. (2.1) are over the narrow band pass considered. The total band absorptance of the so-called "wide band models" is given by

$$A = \int_{-\infty}^{\infty} [1 - \exp(-k_\omega X)] d(\omega - \omega_0) \quad (2.2)$$

where the limits of integration are over the entire band pass and ω_0 is the wave number at the center of the wide band. In actual radiative transfer analyses, the quantity of frequent interest is the derivative of Eqs. (2.1) and (2.2).

Four commonly used narrow band models are Elsasser, Statistical, Random Elsasser, and Quasi-Random. The application of a model to a particular case depends upon the nature of the absorbing emitting molecule. Complete discussion on narrow bands models, and expressions for transmittance and

integrated absorptance are available in the literature [22-26, 31-33]. Detailed discussions on the wide band models are given in [22-26, 34-40]. The relations for total band absorptance of a wide band are obtained from the absorptance formulations of narrow band models by employing the relations for the variation of line intensity as [22-26, 37-40]

$$S_j/d = (S/A_0) \exp\{[-b_0|\omega - \omega_0|]/A_0\} \quad (2.3)$$

where S_j is the intensity of the j th spectral line, d is the line spacing, S is the integrated intensity of a wide band, A_0 is the band width parameter, and $b_0 = 2$ for a symmetrical band and $b_0 = 1$ for bands with upper and lower wave number heads at ω_0 . The total absorptance of an exponential wide band, in turn, may be expressed by

$$\bar{A}(u, \beta) \equiv A(u, \beta)/A_0 = \frac{1}{A_0} \int_{\text{wide band}} [A_N(u, \beta)] d(\omega - \omega_0) \quad (2.4)$$

where $u = SX/A_0$ is the nondimensional path length, $\beta = 2\pi\gamma_L/d$ is the line structure parameter, γ_L is the Lorentz line half-width, and $\bar{A}_N(u, \beta)$ represents the mean absorptance of a narrow band.

By employing the Elsasser narrow band absorptance relation and Eq. (2.3) the expression for the exponential wide band absorptance is obtained as [25,16]

$$\bar{A}(u, \beta) = \gamma + (1/\pi) \int_0^\pi [\ln \psi + E_1(\psi)] dz \quad (2.5)$$

where $\psi = u \sinh \beta / (\cosh \beta - \cos z)$, $\gamma = 0.5772156$ is the Euler's constant, and $E_1(\psi)$ is the exponential integral of the first order. Analytic solution of Eq. (2.5) can be obtained in a series form as [25, 26]

$$\bar{A}(u, \beta) = \sum_{n=1}^{\infty} \{-(A)^n [\text{SUM}(mn)] / [n(B+1)^n n! (n-1)!]\} \quad (2.6)$$

where

$$\text{SUM } (mn) = \sum_{m=0}^{\infty} [(n+m-1)!(2m-1)!C^m]/(2^m(m!)^2]$$

$$A = -u \tanh\beta, \quad B = 1/\cosh\beta,$$

$$C = 2/(1+\cosh\beta) = 2B/(B+1).$$

The series in Eq. (2.6) converges rapidly. When the weak line approximation for the Elsasser model is valid (i.e. β is large), then Eq. (2.5) reduces to

$$\bar{A}(u) = \gamma + \ln(u) + E_1(u). \quad (2.7)$$

In the linear limit, Eqs. (2.5) and (2.6) reduce to $\bar{A} = u$, and in the logarithmic limit they reduce to $\bar{A} = \gamma + \ln(u)$. It can be shown that Eq. (2.5) reduces to the correct limiting form in the square-root limit. Results of Eqs. (2.5) and (2.6) are found to be identical for all pressures and pathlengths. For $p > 1$ atm, results of Eqs. (2.5)-(2.7) are in good agreement for all path lengths.

By employing the uniform statistical, general statistical, and random Elsasser narrow band models absorptance relations and Eq. (2.3), three additional expressions for the exponential wide band absorptance were obtained in [25, 26]. The absorptance results of the four wide band models are discussed in detail in [26]. The expression obtained by employing the uniform statistical model also reduces to the relation (2.7) for large β .

Several continuous correlations for the total absorptance of a wide band, which are valid over different values of path length and line structure parameter, are available in the literature. These are discussed, in detail, in [22-26, 37-40] and are presented here in the sequence that they became available in the literature. Most of these correlations are developed to

satisfy at least some of the limiting conditions (nonoverlapping line, linear, weak line, and strong line approximations, and square-root, large pressure, and large path length limits) for the total band absorptance [23-26]. Some of the correlations even have experimental justifications [22,35].

The first correlation for the exponential wide band absorptance (a three piece correlation) was proposed by Edwards et al. [34, 35]. The first continuous correlation was proposed by Tien and Lowder [22], and this is of the form

$$\bar{A}(u, \beta) = \ln(u f(t) \{u+2\} / [u+2f(t)]) + 1 \quad (2.8)$$

where

$$f(t) = 2.94[1 - \exp(-2.60t)], \quad t = \beta/2.$$

This correlation does not reduce to the correct limiting form in the square-root limit [23,26], and its use should be made for $\beta > 0.1$. Another continuous correlation was proposed by Goody and Belton [39], and in terms of the present nomenclature, this is given by

$$\bar{A}(u, \beta) = 2 \ln\{1 + u/[4 + (\pi u/4t)]^{1/2}\}, \quad \beta = 2t. \quad (2.9)$$

Use of this correlation is restricted to relatively small β values [23-26]. Tien and Ling [40] have proposed a simple two parameter correlation for $\bar{A}(u, \beta)$ as

$$\bar{A}(u) = \sinh^{-1}(u) \quad (2.10)$$

which is valid only for the limit of large β . A relatively simple continuous correlation was introduced by Cess and Tiwari [23], and this is of the form

$$\bar{A}(u, \beta) = 2 \ln(1+u/\{2+[u(1+1/\bar{\beta})]^{1/2}\}) \quad (2.11)$$

where $\bar{\beta} = 4t/\pi = 2\beta/\pi$. By slightly modifying Eq. (2.11), another form of the wide band absorptance is obtained as [25, 26]

$$\bar{A}(u, \beta) = 2 \ln(1+u/\{2+[u(c+\pi/2\beta)]^{1/2}\}) \quad (2.12)$$

where

$$c = \begin{cases} 0.1, & \beta < 1 \text{ and all } u \text{ values} \\ 0.1, & \beta > 1 \text{ and } u < 1 \\ 0.25, & \beta > 1 \text{ and } u > 1. \end{cases}$$

Equations (2.11) and (2.12) reduce to all the limiting forms [23]. Based on the formulations of slab band absorptance, Edwards and Balakrishnan [37] have proposed the correlation

$$\bar{A}(u) = \ln(u) + E_1(u) + \gamma + \frac{1}{2} - E_3(u) \quad (2.13)$$

which is valid for large β . For present application, this correlation should be modified by using the technique discussed in [25, 26]. Based upon the formulation of the total band absorptance from the general statistical model, Felske and Tien [38] have proposed a continuous correlation for $\bar{A}(u, \beta)$ as

$$\begin{aligned} \bar{A}(u, \beta) = & 2E_1(t\rho_u) + E_1(\rho_u/2) - E_1[(\rho_u/2)(1+2t)] \\ & + \ln[t\rho_u]^2/(1+2t) + 2\gamma \end{aligned} \quad (2.14)$$

where

$$\rho_u = \{(t/u)[1 + (t/u)]\}^{-1/2}$$

The absorptance relation given by Eq. (2.7) is another simple correlation which is valid for all path lengths and for $t = (\beta/2) > 1$. The relation of Eq. (2.6) can be treated as another correlation applicable to gases whose spectral behavior can be described by the Elsasser model. In [26] Tiwari has shown that the Elsasser as well as random band model formulations for the total band absorptance reduce to Eq. (2.7) for $t > 1$.

Band absorptance results of various correlations are compared and discussed in some detail in [25, 26, 41]. It was found that results of these correlations could be in error by as much as 40% when compared with the exact solutions based on different band models. Felske and Tien's correlation was found to give the least error when compared with the exact solution based on the general statistical model while Tien and Lowder's correlation gave the least error when compared with the exact solution based on the Elsasser model. The results of Cess and Tiwari's correlations followed the trend of general statistical model. Tiwari and Batki's correlation [Eq. 2.6 or 2.7] was found to provide a uniformly better approximation for the total band absorptance at relatively high pressures. The sole motivation in presenting the various correlations here is to see if their use in actual radiative processes made any significant difference in the final results.

In reference 41, use of several continuous correlations for total band absorptance was made to two problems to investigate their influence on the final results of actual radiative processes. For the case of radiative transfer in a gas with internal heat source, it was found that actual center-line temperature results obtained by using the different correlations follow the same general trend as the results of total band absorptance by these correlations. From these results, it may be concluded that use of the Tien and Lowder's correlation should be avoided at lower pressures, but its use is

justified (at moderate and high pressures) to gases whose spectral behavior can be described by the regular Elsasasser band model. For all pressures and path length conditions, use of the Cess and Tiwari's correlations could be made to gases with bands of highly overlapping lines. In a more realistic problem involving flow of an absorbing emitting gas, results of different correlations (except the Tien and Lowder's correlation) differ from each other by less than 6% for all pressures and path lengths. Use of Tien and Lowder's correlations is justified for gases like CO at moderate and high pressures. For gases like CO₂, use of any other correlation is recommended. While Felske and Tien's correlation is useful for all pressures and path lengths to gases having random band structure. Tiwari and Batki's simple correlation could be employed to gases with regular or random band structure but for $P > 1.0$ atm.

3. RADIATIVE FLUX EQUATIONS

For many engineering and astrophysical applications, the radiative transfer equations are formulated for one-dimensional planar systems (Fig. 3.1). For diffuse boundaries and in the absence of scattering, expressions for the radiative flux and its derivative are given as [8]

$$q_{R\lambda}(\tau_\lambda) = 2 B_{1\lambda} E_3(\tau_\lambda) - 2 B_{2\lambda} E_3(\tau_{0\lambda} - \tau_\lambda) + 2 \left[\int_0^{\tau_\lambda} e_{b\lambda}(t) E_2(\tau_\lambda - t) dt - \int_{\tau_\lambda}^{\tau_{0\lambda}} e_{b\lambda}(t) E_2(t - \tau_\lambda) dt \right] \quad (3.1)$$

and

$$-\frac{dq_{R\lambda}}{d\tau_\lambda} = 2 B_{1\lambda} E_2(\tau_\lambda) + 2 B_{2\lambda} E_2(\tau_{0\lambda} - \tau_\lambda) + 2 \int_0^{\tau_{0\lambda}} e_{b\lambda}(t) E_1(|\tau_\lambda - t|) dt \quad (3.2)$$

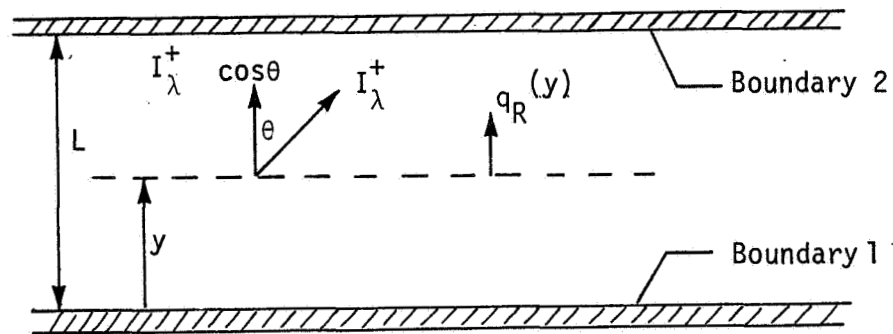


Figure 3.1 Plane radiating layer between parallel boundaries.

where

$$\tau_\lambda = \int_0^y k_\lambda dy, \quad \tau_{0\lambda} = \int_0^L k_\lambda dy \quad (3.3a)$$

$$E_n(t) = \int_0^1 \mu^{n-2} e^{-t/\mu} d\mu \quad (3.3b)$$

In the preceding equations, $E_n(t)$ are the exponential integral functions, and τ_λ and $\tau_{0\lambda}$ represent the optical coordinate and optical path, respectively. The quantities $B_{1\lambda}$ and $B_{2\lambda}$ represent the spectral surface radiosities and for nonreflecting surfaces, $B_{1\lambda} = e_{1\lambda} = \epsilon_{1\lambda} e_{b1\lambda}$ etc. Thus, for non-reflecting boundaries, Eqs. (3.1) and (3.2) are expressed in terms of the wave number as (see Appendix A)

$$q_{R\omega}(\tau_\omega) = e_{1\omega} - e_{2\omega} + 2 \left[\int_0^{\tau_\omega} F_{1\omega}(t) E_2(\tau_\omega - t) dt - \int_{\tau_\omega}^{\tau_{0\omega}} F_{2\omega}(t) E_2(t - \tau_\omega) dt \right] \quad (3.4)$$

and

$$-\frac{dq_{R\omega}}{d\tau_\lambda} = -2 [F_{1\omega}(\tau_\omega) + F_{2\omega}(\tau_\omega)] + 2 \left[\int_0^{\tau_\omega} F_{1\omega}(t) E_1(\tau_\omega - t) dt + \int_{\tau_\omega}^{\tau_{0\omega}} F_{2\omega}(t) E_1(t - \tau_\omega) dt \right] \quad (3.5)$$

where

$$F_{1\omega}(t) = e_\omega(t) - e_{1\omega}; \quad F_{2\omega}(t) = e_\omega(t) - e_{2\omega}$$

Equations (3.4) and (3.5) are the general equations for one-dimensional absorbing-emitting medium with diffuse non-reflecting boundaries. For nongray analyses, it is often convenient to replace the exponential integrals by

appropriate exponential functions [6, 8]. Upon employing the exponential kernal approximation [8]

$$E_2(t) = \frac{3}{4} \exp\left(-\frac{3}{2} t\right); E_1(t) = \frac{9}{8} \exp\left(-\frac{3}{2} t\right)$$

Eqs. (3.4) and (3.5) are expressed in physical coordinatates as

$$\begin{aligned} q_{R\omega}(y) = & e_{1\omega} - e_{2\omega} \\ & + \frac{3}{2} \int_0^y F_{1\omega}(z) k_{\omega} \exp\left[-\frac{3}{2} k_{\omega}(y-z)\right] dz \\ & - \frac{3}{2} \int_y^L F_{2\omega}(z) k_{\omega} \exp\left[-\frac{3}{2} k_{\omega}(z-y)\right] dz \end{aligned} \quad (3.6)$$

$$\begin{aligned} -\frac{dq_{R\omega}}{dy} = & -2 [F_{1\omega}(y) + F_{2\omega}(y)] \\ & + \frac{9}{4} \int_0^y F_{1\omega}(z) k_{\omega}^2 \exp\left[-\frac{3}{2} k_{\omega}(y-z)\right] dz \\ & + \frac{9}{4} \int_y^L F_{2\omega}(z) k_{\omega}^2 \exp\left[-\frac{3}{2} k_{\omega}(z-y)\right] dz \end{aligned} \quad (3.7)$$

where z is a dummy variable for y . However, by differentiating Eq. (3.6) directly, there is obtained

$$\begin{aligned} -\frac{dq_{R\omega}}{dy} = & -\frac{3}{2} k_{\omega} [F_{1\omega}(y) + F_{2\omega}(y)] \\ & + \frac{9}{4} \int_0^y F_{1\omega}(z) k_{\omega}^2 \exp\left[-\frac{3}{2} k_{\omega}(y-z)\right] dz \\ & + \frac{9}{4} \int_y^L F_{2\omega}(z) k_{\omega}^2 \exp\left[-\frac{3}{2} k_{\omega}(z-y)\right] dz \end{aligned} \quad (3.8)$$

The slight difference in Eqs. (3.7) and (3.8) should be noted. This is a consequence of using the exponential kernel approximation. If one has to make a decision as which equation to use, it is recommended to use Eq. (3.8).

The total band absorptance, as given by Eq. (2.2), can be expressed in a slightly different form as

$$A(y) = \int_0^{\infty} [1 - \exp(-k_{\omega} y)] d\omega \sim \text{cm}^{-1} \quad (3.9a)$$

where both k_{ω} and ω have units of cm^{-1} . Differentiation of Eq. (3.9a) gives

$$A'(y) = \int_0^{\infty} k_{\omega} \exp(-k_{\omega} y) d\omega \sim \text{cm}^{-2} \quad (3.9b)$$

and

$$A''(y) = \int_0^{\infty} -k_{\omega}^2 \exp(-k_{\omega} y) d\omega \sim \text{cm}^{-3} \quad (3.9c)$$

Equations (3.9) are employed to express Eqs. (3.6) and (3.8) in terms of the band absorptance.

The total radiative flux is given by

$$q_R(y) = \int_0^{\infty} q_{R\omega}(y) d\omega \quad (3.10)$$

such that

$$\frac{dq_R(y)}{dy} = \int_0^{\infty} \frac{dq_{R\omega}}{dy} d\omega = \frac{d}{dy} \int_0^{\infty} q_{R\omega} d\omega \quad (3.11)$$

Upon substituting Eq. (3.6) into Eq. (3.10) and Eq. (3.8) into Eq. (3.11) there is obtained for a multiband gaseous system

$$q_R(y) = e_1 - e_2$$

$$+ \frac{3}{2} \sum_{i=1}^n \int_{\Delta\omega_i} \left\{ \int_0^y F_{1\omega_i}(z) k_{\omega_i} \exp\left[-\frac{3}{2} k_{\omega_i}(y-z)\right] dz \right. \\ \left. - \int_y^L F_{2\omega_i}(z) k_{\omega_i} \exp\left[-\frac{3}{2} k_{\omega_i}(z-y)\right] dz \right\} d\omega_i \quad (3.12)$$

$$- \frac{dq_R(y)}{dy} = - \frac{3}{2} \sum_{i=1}^n \int_{\Delta\omega_i} k_{\omega_i} [F_{1\omega_i}(y) + F_{2\omega_i}(y)] d\omega_i \\ + \frac{9}{4} \sum_{i=1}^n \int_{\Delta\omega_i} \left\{ \int_0^y F_{1\omega_i}(z) k_{\omega_i}^2 \exp\left[-\frac{3}{2} k_{\omega_i}(y-z)\right] dz \right. \\ \left. + \int_y^L F_{2\omega_i}(z) k_{\omega_i}^2 \exp\left[-\frac{3}{2} k_{\omega_i}(z-y)\right] dz \right\} d\omega_i \quad (3.13)$$

It should be pointed out that the following relations have been used in obtaining Eqs. (3.12) and (3.13)

$$\int_0^\infty e_{1\omega} d\omega = e_1; \quad \int_0^\infty e_{2\omega} d\omega = e_2$$

$$\int_0^\infty \left\{ \int_0^y F_{1\omega}(z) k_{\omega} \exp\left[-\frac{3}{2} k_{\omega}(y-z)\right] dz \right\} d\omega \\ = \sum_{i=1}^n \int_{\Delta\omega_i} \left\{ \int_0^y F_{1\omega_i}(z) k_{\omega_i} \exp\left[-\frac{3}{2} k_{\omega_i}(y-z)\right] dz \right\} d\omega_i$$

where n represents the number of bands in a multiband system.

By utilizing the definitions of the band absorptance and its derivatives as given by Eqs. (3.9) and evaluating the value of the Planck function at the center of each band, Eqs. (3.12) and (3.13) are expressed as

$$\begin{aligned}
 q_R(y) = & e_1 - e_2 \\
 & + \frac{3}{2} \sum_{i=1}^n \left\{ \int_0^y F_{1\omega_{oi}}(z) A_i' \left[\frac{3}{2} (y-z) \right] dz \right. \\
 & \quad \left. - \int_y^L F_{2\omega_{oi}}(z) A_i' \left[\frac{3}{2} (z-y) \right] dz \right\}
 \end{aligned} \tag{3.14}$$

$$\begin{aligned}
 \frac{dq_R(y)}{dy} = & \frac{3}{2} \sum_{i=1}^n \left\{ [F_{1\omega_{oi}}(y) + F_{2\omega_{oi}}(y)] \int_{\Delta\omega_i} k_{\omega_i} d\omega_i \right\} \\
 & + \frac{9}{4} \sum_{i=1}^n \left\{ \int_0^y F_{1\omega_{oi}}(z) A_i'' \left[\frac{3}{2} (y-z) \right] dz \right. \\
 & \quad \left. + \int_y^L F_{2\omega_{oi}}(z) A_i'' \left[\frac{3}{2} (z-y) \right] dz \right\}
 \end{aligned} \tag{3.15}$$

where ω_{oi} represents the center of the i th band.

Equations (3.14) and (3.15) are in proper form for obtaining the nongray solutions of molecular species. However, in order to be able to use the band model correlations, these equations must be transformed in terms of the correlation quantities defined in Eq. (2.4). The following quantities, therefore, are needed for the transformation

$$u = (S/A_0) py; u_0 = (S/A_0) PL; PS = \int_{\Delta\omega} k_{\omega} d\omega \tag{3.16}$$

Now, by using the definition $\bar{A} = A/A_0$, Eq. (3.9b) is written as

$$\bar{A}'(y) = \frac{d}{dy} [\bar{A}(y)] = \frac{1}{A_0} \frac{dA(y)}{dy} = A'(y)/A_0 \sim \text{cm}^{-1}$$

Thus,

$$A'(y) = A_0 \frac{d\bar{A}(y)}{dy} = A_0 \left[\frac{d\bar{A}(u)}{du} \frac{du}{dy} \right] = P S(T) \bar{A}'(u) \quad (3.17a)$$

Similarly

$$A''(y) = [P S(T)]^2 (1/A_0) \bar{A}''(u) \quad (3.17b)$$

The dimensions of both sides in Eqs. (3.17a) and (3.17b) agree with the dimensions given in Eqs. (3.9b) and (3.9c). By employing the definitions of Eqs. (3.16) and (3.17), Eqs. (3.14) and (3.15) are expressed as

$$\begin{aligned} q_R(u) = & e_1 - e_2 \\ & + \frac{3}{2} \sum_{i=1}^n A_{0i} \left\{ \int_0^{u_i} F_{1\omega_i}(u'_i) \bar{A}'_i \left[\frac{3}{2} (u_i - u'_i) \right] du'_i \right. \\ & \left. - \int_{u_i}^{u_{0i}} F_{2\omega_i}(u'_i) \bar{A}'_i \left[\frac{3}{2} (u'_i - u_i) \right] du'_i \right\} \end{aligned} \quad (3.18)$$

$$\begin{aligned} \frac{dq_R(u)}{du} = & \frac{3}{2} \sum_{i=1}^n A_{0i} [F_{1\omega_i}(u) + F_{2\omega_i}(u)] \\ & + \frac{9}{4} \sum_{i=1}^n A_{0i} \left\{ \int_0^{u_i} F_{1\omega_i}(u') \bar{A}''_i \left[\frac{3}{2} (u_i - u'_i) \right] du'_i \right. \\ & \left. + \int_{u_i}^{u_{0i}} F_{2\omega_i}(u') \bar{A}''_i \left[\frac{3}{2} (u'_i - u_i) \right] du'_i \right\} \end{aligned} \quad (3.19)$$

where u' is the dummy variable for u and $\bar{A}'(u) = d\bar{A}/du$. It should be noted that $F_{1\omega_i}$ and $F_{2\omega_i}$ in Eqs. (3.18) and (3.19) represent the values of

$F_{1\omega}$ and $F_{2\omega}$ at the center of the i th band, and $dq_R/dy = (dq_R/du)(du/dy) = [P S(T)/A_0] (dq_R/du)$.

By defining the new independent variables as

$$\xi = u/u_0 = y/L; \quad \xi' = u'/u_0 = z/L \quad (3.20)$$

Eqs. (3.18) and (3.19) can be expressed as

$$\begin{aligned} q_R(\xi) = & e_1 - e_2 \\ & + \frac{3}{2} \sum_{i=1}^n A_{oi} u_{oi} \left\{ \int_0^{\xi} F_{1\omega_i}(\xi') \bar{A}'_i \left[\frac{3}{2} u_{oi} (\xi - \xi') \right] d\xi' \right. \\ & \left. - \int_{\xi}^1 F_{2\omega_i}(\xi') \bar{A}'_i \left[\frac{3}{2} u_{oi} (\xi' - \xi) \right] d\xi' \right\} \end{aligned} \quad (3.21)$$

$$\begin{aligned} \frac{dq_R(\xi)}{d\xi} = & \frac{3}{2} \sum_{i=1}^n \{ [F_{1\omega_i}(\xi) + F_{2\omega_i}(\xi)] (A_{oi} u_{oi}) \} \\ & + \frac{9}{4} \sum_{i=1}^n A_{oi} u_{oi}^2 \left\{ \int_0^{\xi} F_{1\omega_i}(\xi') \bar{A}''_i \left[\frac{3}{2} u_{oi} (\xi - \xi') \right] d\xi' \right. \\ & \left. + \int_{\xi}^1 F_{2\omega_i}(\xi') \bar{A}''_i \left[\frac{3}{2} u_{oi} (\xi' - \xi) \right] d\xi' \right\} \end{aligned} \quad (3.22)$$

where again $\bar{A}'(u)$ denotes the derivative of $A(u)$ with respect to u , and $dq_R/du = (dq_R/d\xi) (d\xi/du) = (1/u_0) (dq_R/d\xi)$.

Equations (3.18) through (3.22) allow us to make use of the band model correlations for the wide-band absorptance because these correlations are expressed in terms of u and β . However, it is often desirable and convenient to express the relations for q_R and $\text{div } q_R$ which only involve $\bar{A}(u)$ and $\bar{A}'(u)$ but not $\bar{A}''(u)$. This is accomplished by integrating the

integrals in the expressions for q_R and $\text{div } q_R$ by parts. This results in simpler integrals. Upon performing the integration by parts on the integrals in Eqs. (3.21) and (3.22), the equations can be expressed in alternate forms as (see Appendix B)

$$\begin{aligned}
 q_R(\xi) = & e_1 - e_2 \\
 & + \sum_{i=1}^n A_{oi} \left\{ \int_0^{\xi} [de_{\omega_i}(\xi')/d\xi'] \bar{A}_i \left[\frac{3}{2} u_{oi}(\xi - \xi') \right] d\xi' \right. \\
 & \left. + \int_{\xi}^1 [de_{\omega_i}(\xi')/d\xi'] \bar{A}_i \left[\frac{3}{2} u_{oi}(\xi' - \xi) \right] d\xi' \right\} \quad (3.23)
 \end{aligned}$$

$$\begin{aligned}
 \frac{dq_R(\xi)}{d\xi} = & \frac{3}{2} \sum_{i=1}^n A_{oi} u_{oi} \left\{ \int_0^{\xi} [de_{\omega_i}(\xi')/d\xi'] \bar{A}_i' \left[\frac{3}{2} u_{oi}(\xi - \xi') \right] d\xi' \right. \\
 & \left. - \int_{\xi}^1 [de_{\omega_i}(\xi')/d\xi'] \bar{A}_i' \left[\frac{3}{2} u_{oi}(\xi' - \xi) \right] d\xi' \right\} \quad (3.24)
 \end{aligned}$$

It should be noted that Eq. (3.24) can be obtained directly by differentiating Eq. (3.23) with respect to ξ using the Leibnitz formula. This is shown in Appendix B.

Equations (3.21), (3.23), and (3.24) are the most convenient equations to use when employing the band-model correlations in radiative transfer analyses.

4. BASIC FORMULATION FOR TRANSIENT PROCESSES

The interaction of radiation in transient transfer processes has received very little attention in the literature. Yet, the transient approach appears to be the logical way of formulating a problem in general sense for elegant numerical and computational solutions. The steady-state solutions can be obtained as limiting solutions for large times.

A few studies available on radiative interactions reveal that the transient behavior of a physical system can be influenced significantly in the presence of radiation [42-45]. Lick investigated the transient energy transfer by radiation and conduction through a semi-finite medium [42]. A kernel substitution technique was used to obtain analytic solutions and display the main features and parameters of the problem. Doornink and Hering studied the transient radiative transfer in a stationary plane layer of a nonconducting medium bounded by black walls [43]. A rectangular Milne-Eddington type relation was used to describe the frequency dependence of the absorption coefficient. It was found that the cooling of the layer initially at a uniform temperature is strongly dependent on the absorption coefficient model employed. Larson and Viskanta investigated the problem of transient combined laminar free convection and radiation in a rectangular enclosure [44]. It was demonstrated that the radiation dominates the heat transfer in the enclosure and alters the convective flow patterns significantly. The transient heat exchange between a radiation plate and a high-temperature gas flow was investigated by Melnikov and Sukhovich [45]. Only the radiative interaction from the plate was considered; the gas was treated as a non-participating medium. It was proved that the surface temperature is a function of time and of longitudinal coordinate.

The objective of this study is to investigate the interaction of nongray radiation in transient transfer processes in a general sense. Attention, however, will be directed first to a simple problem of the transient radiative exchange between two parallel plates. In subsequent studies, the present analysis and numerical techniques will be extended to include the flow of homogeneous, nonhomogeneous, and chemically reacting species in one- and multi-dimensional systems.

The physical model considered for the present study is the transient energy transfer by radiation in absorbing-emitting gases bounded by two parallel gray plates (Fig. 4.1). In general, T_1 and T_2 can be a function of time and position and there may exist an initial temperature distribution in the gas. It is assumed that the radiative energy transfer in the axial direction is negligible in comparison to that in the normal direction.

For radiation participating medium, the equations expressing conservation of mass and momentum remain unaltered, while the conservation of energy, in general, is expressed as [8]

$$\rho c_p \frac{DT}{Dt} = \text{div} (k \text{ grad } T) + \beta T \frac{DP}{Dt} + \mu \phi - \text{div } q_R \quad (4.1)$$

where β is the coefficient of thermal expansion of the fluid and ϕ is the Rayleigh dissipation function. For a semi-infinite medium capable of transferring energy only by radiation and conduction, Eq. (4.1) reduces to

$$\rho c_p \frac{\partial T}{\partial t} = - \frac{\partial q}{\partial y} \quad (4.2)$$

where q is the sum of the conductive heat flux $q_c = -k (\partial T / \partial y)$ and the radiative flux q_R . For the physical model where radiation is the only mode of energy transfer, the energy equation can be written as

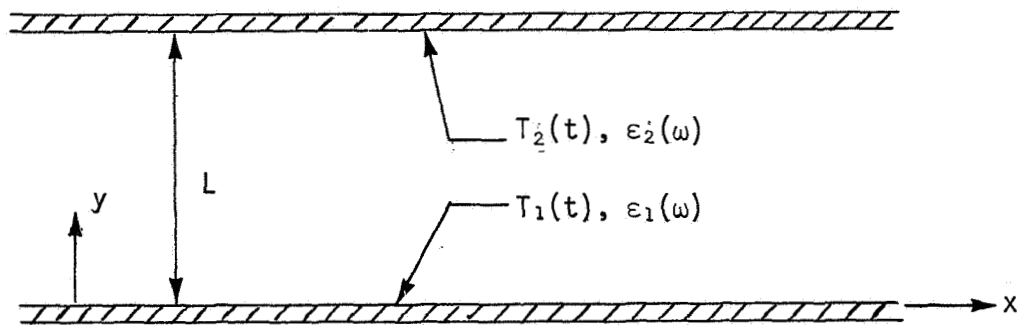


Figure 4.1 Physical model and coordinate system.

$$\rho c_p \frac{\partial T}{\partial t} = - \frac{\partial q_R}{\partial y} \quad (4.3)$$

Use of this simplified equation is made to investigate the transient behavior of a radiation participating medium.

As pointed out in the previous section (Sec. 3), Eqs. (3.21) and (3.23) are convenient equations for the radiative flux. Equations (3.22) and (3.24) are two expressions for the $\text{div } q_R(y)$, but Eq. (3.24) is preferred because it only involves the first derivative of \bar{A} and avoids singularities in the large path length limit.

Upon defining nondimensional radiative heat flux by

$$Q(\xi, t) = q_R(\xi, t) / [e_1(t) - e_2(t)] \quad (4.4)$$

Eq. (3.21) can be written as

$$Q(\xi, t) = 1 + \frac{3}{2} \sum_{i=1}^n A_{oi} u_{oi} \left\{ \int_0^{\xi} \zeta_{1i}(\xi', t) \bar{A}'_i \left[\frac{3}{2} u_{oi}(\xi - \xi') \right] d\xi' - \int_{\xi}^1 \zeta_{2i}(\xi', t) \bar{A}'_i \left[\frac{3}{2} u_{oi}(\xi' - \xi) \right] d\xi' \right\} \quad (4.5)$$

where

$$\zeta_i(\xi, t) = F_{\omega i}(\xi, t) / [e_1(t) - e_2(t)]$$

Equation (4.5) provides the general expression for the radiative flux in the nondimensional form. A similar nondimensional form can be obtained also from Eq. (3.23).

By defining $\phi(\xi, t) = T(\xi, t) / T_0$ with T_0 representing some constant reference temperature, Eqs. (4.3) and (3.24) can be combined to yield the energy equation in nondimensional form as

$$\begin{aligned}
-\partial \phi(\xi, t) / \partial t = & \frac{9}{4} \sum_{i=1}^n \left\{ \int_0^{\xi} \psi_{\omega i}(\xi', t) \bar{A}_i' \left[\frac{3}{2} u_{oi}(\xi - \xi') \right] d\xi' \right. \\
& \left. - \int_{\xi}^1 \psi_{\omega i}(\xi', t) \bar{A}_i' \left[\frac{3}{2} u_{oi}(\xi' - \xi) \right] d\xi' \right\} \quad (4.6)
\end{aligned}$$

where

$$\psi_{\omega i}(\xi, t) = \{PS_i(T)[\partial e_{\omega i}(\xi, t) / \partial \xi] / (\rho C_p T_0 / t_m)\}$$

The time t in Eq. (4.6) is defined as $t^* = t/t_m$ with t_m representing some characteristic time scale of the physical problem; however, for the sake of convenience, the asterisk is left out here as well as in further developments. From the definitions of $\phi(\xi, t)$ and $\psi_{\omega i}(\xi, t)$, it should be noted that Eq. (4.6) is a nonlinear equation in $T(\xi, t)$. Equation (4.6), therefore, represents a general case of the transient energy by radiation between two semi-infinite parallel plates. A similar expression can be obtained also by combining Eqs. (4.3) and (3.22).

5. A SPECIAL CASE OF TRANSIENT INTERACTION

As a special case, it is assumed that the entire system initially is at the fixed (reference) temperature T_0 . For all time, the temperature of the upper plate is maintained at the constant temperature equal to the reference temperature, i.e., $T_2 = T_0$. The temperature of the lower plate is suddenly decreased to a lower but constant temperature, i.e., $T_1 < T_2$. The problem, therefore, is to investigate the transient cooling rate of the gas for a step change in temperature of the lower plate.

Since small temperature differences have been assumed and the absorption coefficient has been taken as independent of temperature one may employ additionally the linearization,

$$e_{\omega i}(T) - e_{\omega i}(T_w) \approx (d e_{\omega i}/dT)_{T_w} (T - T_w) \quad (5.1)$$

where again the subscript i refers to the i th band such that ω_i is the wave number location of the band and T_w represents the temperature of the reference wall which could be either T_1 or T_2 . For the special case considered, since we are interested in investigating the transient behavior of the gas because of a step change in temperature of the lower plate, T_w is taken to be equal to T_1 . Thus,

$$e_{\omega i}(\xi, t) - e_{\omega i}(0, t) = (d e_{\omega i}/dT)_{T_1} (T - T_1) \quad (5.2a)$$

$$e_{\omega i}(1, t) - e_{\omega i}(0, t) = (d e_{\omega i}/dT)_{T_1} (T_2 - T_1) \quad (5.2b)$$

$$e_{\omega i}(\xi, t) - e_{\omega i}(1, t) = (d e_{\omega i}/dT)_T (T - T_2) \quad (5.2c)$$

Note that Eq. (5.2c) is obtained by subtracting Eq. (5.2b) from Eq. (5.2a). Also, for linearized radiation,

$$T^4 = 4 T_1^3 T - 3 T_1^4 \quad (5.3)$$

Thus, $e_1 = \sigma T_1^4$, $e_2 = \sigma T_2^4 = \sigma (4 T_1^3 T_2 - 3 T_1^4)$ such that $e_1 - e_2 = 4 \sigma T_1^3 (T_1 - T_2)$.

It should be pointed out that for a single-band gas, the linearization is not required because the temperature distribution can be obtained either by combining Eqs. (3.22) and (4.3) or from Eq. (4.6) and the radiative heat flux can be calculated from Eqs. (3.21), (3.23), or (4.5). However, for the case of multiband gases and for systems involving mixtures of gases, it is convenient to employ the linearization procedure in order to use the information on band model correlations. The following definitions are useful

in expressing the governing equations in linearized forms:

$$\theta = (T - T_1) / (T_2 - T_1) \quad (5.4a)$$

$$N_{1i} = (P t_m / \rho c_p) K_{1i}, \quad K_{1i} = S_i(T) (d e_{\omega i} / dT)_{T_1} \quad (5.4b)$$

$$N_1 = (P t_m / \rho c_p) K_1, \quad K_1 = \sum_{i=1}^n K_{1i} \quad (5.4c)$$

$$M_{1i} = (t_m / L \rho c_p) H_{1i}, \quad H_{1i} = A_{oi}(T) (d e_{\omega i} / dT)_{T_1} \quad (5.4d)$$

$$M_1 = (t_m / L \rho c_p) H_1, \quad H_1 = \sum_{i=1}^n H_{1i} \quad (5.4e)$$

$$M_{1i} u_{oi} = N_{1i}, \quad u_{oi} H_{1i} = PL K_{1i} \quad (5.4f)$$

where H_1 , K_1 , N_1 and M_1 represent the values of H , K , N and M evaluated at the temperature T_1 . As explained in Refs. 8 and 23, these quantities represent the properties of the gaseous medium.

By employing the definitions of Eqs. (5.2) - (5.4), relations for the radiative flux, as given by Eqs. (3.21) and (3.23), are expressed as

$$\begin{aligned} Q(\xi, t) = 1 - (3/8 \sigma T_1^3) \sum_{i=1}^n u_{oi} H_{1i} \left\{ \int_0^\xi \theta(\xi', t) \bar{A}_i \left[\frac{3}{2} u_{oi} (\xi - \xi') \right] d\xi' \right. \\ \left. + \int_\xi^1 [1 - \theta(\xi', t)] \bar{A}_i \left[\frac{3}{2} u_{oi} (\xi' - \xi) \right] d\xi' \right\} \end{aligned} \quad (5.5a)$$

and

$$Q(\xi, t) = 1 - (1/4 \sigma T_1^3) \sum_{i=1}^n H_{1i} \left\{ \int_0^\xi \frac{\partial \theta(\xi', t)}{\partial \xi'} \bar{A}_i \left[\frac{3}{2} u_{oi} (\xi - \xi') \right] d\xi' \right.$$

$$- \int_{\xi}^1 \frac{\partial \theta(\xi', t)}{\partial \xi'} \bar{A}_i \left[\frac{3}{2} u_{oi} (\xi' - \xi) \right] d\xi' \quad (5.5b)$$

Thus, the expressions for the heat flux at the lower wall are given by

$$Q(0, t) = 1 - (3/8 \sigma T_1^3) \sum_{i=1}^n u_{oi} H_{1i} \int_0^1 [1 - \theta(\xi', t)] \bar{A}_i \left(\frac{3}{2} u_{oi} \xi' \right) d\xi' \quad (5.6a)$$

and

$$Q(0, t) = 1 + (1/4 \sigma T_1^3) \sum_{i=1}^n H_{1i} \int_0^1 \frac{\partial \theta(\xi', t)}{\partial \xi'} \bar{A}_i \left(\frac{3}{2} u_{oi} \xi' \right) d\xi' \quad (5.6b)$$

It should be pointed out that Eqs. (5.5a) and (5.6a) are convenient forms for the optically thin and general solutions while Eqs. (5.5b) and (5.6b) are useful for solutions in the large path length limit. Once the solutions for $\theta(\xi, t)$ are known from the energy equation, the appropriate relations for the heat flux can be obtained from Eqs. (5.5) and (5.6).

By employing the definitions of Eqs. (5.2) - (5.4), a combination of Eqs. (3.22) and (4.3) provides one form of the energy equation and Eq. (4.6) is transformed to obtain another form; these are expressed as

$$\begin{aligned} \frac{\partial \theta(\xi, t)}{\partial t} + 3 N_1 \theta(\xi, t) - \frac{3}{2} N_1 &= \\ &= - \frac{9}{4} \sum_{i=1}^n M_{1i} u_{oi}^2 \left\{ \int_0^{\xi} \theta(\xi', t) \bar{A}_i' \left[\frac{3}{2} u_{oi} (\xi - \xi') \right] d\xi' \right. \\ &\quad \left. + \int_{\xi}^1 [\theta(\xi', t) - 1] \bar{A}_i' \left[\frac{3}{2} u_{oi} (\xi' - \xi) \right] d\xi' \right\} \end{aligned} \quad (5.7a)$$

and

$$- \frac{\partial \theta(\xi, t)}{\partial t} = \frac{3}{2} \sum_{i=1}^n N_{1i} \left\{ \int_0^{\xi} \frac{\partial \theta(\xi', t)}{\partial \xi'} \bar{A}_i \left[\frac{3}{2} u_{oi} (\xi - \xi') \right] d\xi' \right.$$

$$- \int_{\xi}^1 \frac{\partial \theta(\xi', t)}{\partial \xi'} \bar{A}'_1 \left[\frac{3}{2} u_{01}(\xi' - \xi) \right] d\xi' \quad (5.7b)$$

The initial and boundary conditions for Eq. (5.7) are specified as

$$\theta(\xi, 0) = 1 ; \theta(0, t) = 0 ; \theta(1, t) = 1 \quad (5.8)$$

The parameters in Eq. (5.7) are N_1 and u_0 . For a given gas, the parameters are the gas pressure and the temperature of the lower wall. Equation (5.7b) is the convenient form for solutions in the large path length limit.

6. METHOD OF SOLUTIONS

For the general case, the temperature distribution is obtained from the solution of the energy equation, Eqs. (5.7). Once $\theta(\xi, t)$ is known, the radiative heat flux is calculated by using the appropriate form of Eq. (5.6). Before discussing the solution procedure for the general case, however, it is desirable to obtain the limiting forms of Eqs. (5.5) and (5.7) in the optically thin and large path length limits and investigate the solutions of resulting equations.

6.1 Optically Thin Limit

In the optically thin limit [8, 23], $\bar{A}(u) = u$, $\bar{A}'(u) = 1$, and $\bar{A}''(u) = 0$. In this limit, therefore, Eq. (5.7a) reduces to

$$\frac{\partial \theta(\xi, t)}{\partial t} + 3 N_1 \theta(\xi, t) - \frac{3}{2} N_1 = 0 \quad (6.1a)$$

From an examination of Eq. (6.1a) along with the definitions given in Eq. (5.4), it is evident that in the optically thin limit the temperature distribution in the medium is independent of the ξ -coordinate. This is a characteristic of the optically thin radiation [8, 23]. Thus, Eq. (6.1a) can

be written as

$$\frac{d\theta(t)}{dt} + 3 N_1(t) \theta(t) - \frac{3}{2} N_1(t) = 0; \theta(\xi, 0) = 1 \quad (6.1b)$$

Since gas properties are evaluated at known reference conditions, N_1 is essentially constant, and solution of Eq. (6.1b) is found to be

$$\theta(t) = \frac{1}{2}[1 + \exp(-3 N_1 t)] \quad (6.2)$$

In the optically thin limit, Eq. (5.7b) reduces to

$$-\frac{\partial \theta(\xi, t)}{\partial t} = \frac{3}{2} \left(\sum_{i=1}^n N_{1i} \right) \left\{ \int_0^{\xi} \frac{\partial \theta(\xi', t)}{\partial \xi'} d\xi' - \int_{\xi}^1 \frac{\partial \theta(\xi', t)}{\partial \xi'} d\xi' \right\} \quad (6.3a)$$

A differentiation of Eq. (6.3a) with respect to ξ (by using the Leibnitz's rule) results in

$$\frac{\partial}{\partial \xi} \left[\frac{\partial \theta(\xi, t)}{\partial t} + 3 N_1 \theta(\xi, t) \right] = 0 \quad (6.3b)$$

or

$$\frac{\partial \theta(\xi, t)}{\partial t} + 3 N_1 \theta(\xi, t) = C(t) \quad (6.3c)$$

The constant of integration $C(t)$ is evaluated through the combination of Eqs. (5.8) and (6.3a) and is found to be $C(t) = \frac{3}{2} N_1$. A substitution of this in Eq. (6.3c) gives Eq. (6.1a) for which the solution is given by Eq. (6.2). Thus, as would be expected, both general forms of the energy equation reduce to the same equation in the optically thin limit.

In the optically thin limit Eqs. (5.5a) and (5.5b) respectively reduce to

$$Q(\xi, t) = 1 - [3/(8\sigma T_1^3)] (PLK_1) \left\{ \int_0^{\xi} \theta(\xi', t) d\xi' + \int_{\xi}^1 [1 - \theta(\xi', t)] d\xi' \right\} \quad (6.4a)$$

and

$$Q(\xi, t) = 1 - [3/8\sigma T_1^3] (PLK_1) \left\{ \int_0^\xi \frac{\partial \theta}{\partial \xi'} (\xi - \xi') d\xi' + \int_\xi^1 \frac{\partial \theta}{\partial \xi'} (\xi' - \xi) d\xi' \right\} \quad (6.4b)$$

Through integration by parts, it can be shown that Eq. (6.4b) reproduces Eq. (6.4a). By noting that, in the optically thin limit, $\theta(\xi, t) = \theta(t)$, Eqs. (6.4) can be expressed as

$$Q(\xi, t) = 1 - [3/(8\sigma T_1^3)] (PLK_1) [(1-\xi) + (2\xi-1) \theta(t)] \quad (6.5)$$

It should be pointed out that Eq. (6.5) can be obtained directly from Eq. (6.4b) without performing the integration by parts. The heat transfer from the lower surface in the optically thin limit, therefore, is given by

$$Q(0, t) = 1 - [3/(8\sigma T_1^3)] (PLK_1) [1 - \theta(t)] \quad (6.6)$$

The result of Eq. (6.6) can be obtained directly by letting $\xi = 0$ in either of Eqs. (6.4). The relation for $\theta(t)$ in Eq. (6.6) is obtained from Eq. (6.2). Thus, evaluation of the temperature distribution and radiative heat flux in the optically thin limit does not require numerical solutions.

6.2 Large Path Length Limit

In the large path length limit (i.e., for $u_{0i} \gg 1$ for each band), one has $\bar{A}(u) = \lambda n(u)$, $\bar{A}'(u) = 1/u$, and $\bar{A}''(u) = -1/u^2$ [8, 23]. Thus, in the large path length limit, Eq. (5.7a) reduces to

$$\begin{aligned} \frac{\partial \theta(\xi, t)}{\partial t} + 3 N_1 \theta(\xi, t) - \frac{3}{2} N_1 \\ = M_1 \left\{ \int_0^\xi \theta(\xi', t) \frac{d\xi'}{(\xi - \xi')^2} + \int_\xi^1 [\theta(\xi', t) - 1] \frac{d\xi'}{(\xi' - \xi)^2} \right\} \end{aligned} \quad (6.7a)$$

It should be noted that for any fixed value T_1 and a given gas, N_1 and M_1 are

constants; but, $\theta(\xi, t)$ does depend on ξ . For a given gas and with known values of T_2 and t_m , the solution of Eq. (6.7a) can be obtained by specifying T_1 . Equation (6.7a) involves singular integrals with Cauchy type kernels and, therefore, a closed form solution does not appear to be possible; numerical solutions, however, can be obtained by the variation of parameter technique. Because of the singular nature of integrals, Eq. (6.7a) is not a convenient equation for the large path length limit solutions.

In the large path length limit, Eq. (5.7b) reduces to

$$\frac{\partial \theta(\xi, t)}{\partial t} = - M_1 \int_0^1 \frac{\partial \theta(\xi', t)}{\partial \xi'} \frac{d\xi'}{(\xi - \xi')} \quad (6.7b)$$

Equation (6.7b) is a convenient form for solution in the large path length limit. An analytical solution of Eq. (6.7b) may be possible, but numerical solution can be obtained quite easily.

In the large path length limit, Eqs. (5.5a) and (5.5b) reduce respectively to

$$Q(\xi, t) = 1 - (1/4T_1^3) H_1 \left[\int_0^1 \theta(\xi', t) \frac{d\xi'}{(\xi - \xi')} - \int_{\xi}^1 \frac{d\xi'}{(\xi - \xi')} \right] \quad (6.8a)$$

and

$$Q(\xi, t) = 1 - (1/4\sigma T_1^3) \sum_{i=1}^n H_{1i} \left\{ \int_0^{\xi} \frac{\partial \theta(\xi', t)}{\partial \xi'} \ln \left[\frac{3}{2} u_{oi}(\xi - \xi') \right] d\xi' \right. \\ \left. + \int_{\xi}^1 \frac{\partial \theta(\xi', t)}{\partial \xi'} \ln \left[\frac{3}{2} u_{oi}(\xi' - \xi) \right] d\xi' \right\} \quad (6.8b)$$

The expressions for dimensionless radiative heat flux from or to the wall are obtained by setting $\xi = 0$ in Eqs. (6.8) as

$$Q(0,t) = 1 - (H_1/4\sigma T_1^3) \int_0^1 [1 - \theta(\xi',t)] \frac{d\xi'}{\xi'} \quad (6.9a)$$

and

$$Q(0,t) = 1 - (1/4\sigma T_1^3) \sum_{i=1}^n H_{1i} \int_0^1 \frac{\partial \theta(\xi',t)}{\partial \xi'} \ln\left(\frac{3}{2} u_{oi} \xi'\right) d\xi' \quad (6.9b)$$

Thus, once the temperature distribution is known from solutions of Eq. (6.7), the wall heat flux can be calculated by using the corresponding form of Eqs. (6.9).

6.3 Numerical Solutions of Governing Equations

General solutions of Eqs. (5.7a) and (5.7b) are obtained numerically by employing the method of variation of parameters. For this, a polynomial form for $\theta(\xi,t)$ is assumed in powers of ξ with time dependent coefficients as

$$\theta(\xi,t) = \sum_{m=0}^n c_m(t) \xi^m \quad (6.10)$$

By considering only the quadratic solution in ξ , and satisfying the boundary conditions of Eq. (5.8), one finds

$$\theta(\xi,t) = \xi^2 + g(t) (\xi - \xi^2) \quad (6.11)$$

where $g(t)$ represents the time dependent coefficient. At $t = 0$, a combination of Eqs. (5.8) and (6.11) yields the result

$$g(0) = (1 - \xi^2)/(\xi - \xi^2) \quad (6.12)$$

Also, from Eq. (6.11) there is obtained

$$\frac{\partial \theta(\xi,t)}{\partial t} = (\xi - \xi^2) \frac{dg(t)}{dt} = (\xi - \xi^2) g'(t) \quad (6.13)$$

and

$$\frac{\partial \theta(\xi, t)}{\partial \xi} = 2 \xi + g(t) (1-2\xi) \quad (6.14)$$

Equations (6.11) - (6.14) are employed to obtain specific solutions of Eqs. (5.5) and (5.7).

By substituting Eqs. (6.11) and (6.13) in Eq. (5.7a), there is obtained

$$g'(t) + G_1(\xi) g(t) = G_2(\xi) \quad (6.15)$$

where the integral functions $G_1(\xi)$ and $G_2(\xi)$ are defined in Appendix C. The solution of Eq. (6.15) is given by

$$g(t) = c \exp[-G_1(\xi)t] + G_2(\xi)/G_1(\xi) \quad (6.16a)$$

Since at $t = 0$, $g(t) = g(0)$, then $c = g(0) - G_2(\xi)/G_1(\xi)$. Thus, Eq. (6.16a) becomes

$$g(t) = [g(0) - G_2(\xi)/G_1(\xi)] \exp[-G_1(\xi)t] + G_2(\xi)/G_1(\xi) \quad (6.16b)$$

where $g(0)$ is given by Eq. (6.12). The integrals in functions $G_1(\xi)$ and $G_2(\xi)$ can be evaluated easily by numerical means, after substituting the relation for $\bar{A}''(u)$.

A substitution of Eqs. (6.11), (6.13) and (6.14) into Eq. (5.7b) results in

$$g'(t) + G_3(\xi) g(t) = G_4(\xi) \quad (6.17)$$

where the integral functions $G_3(\xi)$ and $G_4(\xi)$ are defined in Appendix C. The solution of Eq. (6.17) is found to be

$$g(t) = [g(0) - G_4(\xi)/G_3(\xi)] \exp[-G_3(\xi)t] + G_4(\xi)/G_3(\xi) \quad (6.18)$$

where again $g(0)$ is given by Eq. (6.12).

The solutions of Eqs. (5.7a) and (5.7b) can be expressed in a convenient form as

$$\theta(\xi, t) = \xi^2 + \left[\frac{g_1(t)}{g_2(t)} \right] (\xi - \xi^2) \quad (6.19)$$

In Eq. (6.19), $g_1(t)$ is given by Eq. (6.16b) and is used for the solution of Eq. (5.7a) and $g_2(t)$ is given by Eq. (6.18) and is used in obtaining the solution of Eq. (5.7b). The both approach should result in the same final solution.

For the steady state case, the solution again is given by Eq. (6.19), but functions $g_1(t)$ and $g_2(t)$ are no longer a function of time and are given by

$$g_1 = G_2(\xi)/G_1(\xi); \quad g_2 = G_4(\xi)/G_3(\xi) \quad (6.20)$$

The solutions for the steady case are available in the literature and are useful in comparing the results of this study in the limit of $t \rightarrow \infty$.

The expressions for the nondimensional radiative flux are obtained from a combination of Eqs. (5.5), (6.11) and (6.14) such that

$$Q(\xi, t) = 1 - G_5(\xi) g_1(t) - G_6(\xi) \quad (6.21a)$$

and

$$Q(\xi, t) = 1 - G_7(\xi) g_2(t) - G_8(\xi) \quad (6.21b)$$

where $G_5(\xi)$ through $G_8(\xi)$ are defined in Appendix C, and $q_1(t)$ and $q_2(t)$ are given respectively by Eqs. (6.16b) and 6.18). Consequently, the expressions for the radiative heat flux at the lower wall are obtained as

$$Q(0,t) = 1 + G_9 q_1(t) - G_{10} \quad (6.22a)$$

and

$$Q(0,t) = 1 + G_{11} g_2(t) + G_{12} \quad (6.22b)$$

where G_9 through G_{12} are defined in Appendix C and are not function of ξ . It should be noted that the solutions presented in Eqs. (6.21) and (6.22) require the solution of the energy equation as given by Eqs. (6.19)

6.4 Numerical Solutions of Large Path Length Equations

As mentioned earlier, Eqs. (6.7b), (6.8b) and (6.9b) are the most appropriate equations to use in the large path length limit. However, numerical procedure is presented for both forms of the energy and radiative flux equations. Once again Eqs. (6.11) through (6.14) provide the basis for numerical solutions also in the large path length limit. For this limit, the solution given by Eq. (6.19) is expressed as

$$\theta(\xi,t) = \xi^2 + \left[\frac{g_3(t)}{g_4(t)} \right] (\xi, \xi^2) \quad (6.23)$$

where $g_3(t)$ is used for the solution of Eq. (6.7a) and $g_4(t)$ for Eq. (6.7b).

A substitution of Eq. (6.23) into Eq. (6.7a) results in

$$g_3'(t) + G_{13}(\xi) g_3(t) = G_{14}(\xi) \quad (6.24)$$

where integral functions $G_{13}(\xi)$ and $G_{14}(\xi)$ are defined in Appendix C. The solution of Eq. (6.24) is found to be

$$g_3(t) = [g(0) - G_{14}(\xi)/G_{13}(\xi)] \exp[-G_{13}(\xi)t] + G_{14}(\xi)/G_{13}(\xi) \quad (6.25)$$

where $g(0)$ is defined again by Eq. (6.12). Equation (6.23) along with Eq. (6.25) provides the solution of the energy equation, Eq. (6.7a).

A combination of Eqs. (6.23) and (6.7b) results in

$$g_4'(t) + G_{15}(\xi) g_4(t) = G_{16}(\xi) \quad (6.26)$$

The integral functions $G_{15}(\xi)$ and $G_{16}(\xi)$ appearing in Eq. (6.26) are defined in Appendix C. These, however, can be evaluated easily with the results

$$G_{15}(\xi) = [M_1/(\xi - \xi^2)] \{2 + (2\xi - 1) \ln[(\xi - 1)/\xi]\} \quad (6.27a)$$

$$G_{16}(\xi) = 2 [M_1/(\xi - \xi^2)] \{1 + \xi \ln[(\xi - 1)/\xi]\} \quad (6.27b)$$

The solution of Eq. (6.26) is found to be

$$g_4(t) = [g(0) - G_{16}(\xi)/G_{15}(\xi)] \exp[-G_{15}(\xi)t] + G_{16}(\xi)/G_{15}(\xi) \quad (6.28)$$

where again $g(0)$ is defined by Eq. (6.12). A combination of Eqs. (6.23), (6.27) and (6.28) provides the solution of the energy equation, Eq. (6.7b). The only parameter appearing in the solution of Eq. (6.7b) is M_1 .

The expressions for the nondimensional heat flux in this case is obtained from a combination of Eqs. (6.8) and (6.23) as

$$Q(\xi, t) = 1 - G_{17}(\xi) g_3(t) - G_{18}(\xi) \quad (6.29a)$$

and
$$Q(\xi, t) = 1 - G_{19}(\xi) g_4(t) - G_{20}(\xi) \quad (6.29b)$$

where $G_{17}(\xi)$ through $G_{20}(\xi)$ are defined in Appendix C and can be evaluated in closed forms. The corresponding expressions for the radiative heat flux at the lower wall are found from Eqs. (6.9) and (6.23) as

$$Q(0, t) = 1 - G_{21} g_3(t) - G_{22} \quad (6.30a)$$

and
$$Q(0,t) = 1 - G_{23} g_4(t) - G_{24} \quad (6.30b)$$

where again G_{21} through G_{24} are defined in Appendix C and are not functions of ξ .

7. RADIATIVE INTERACTION IN LAMINAR FLOWS

The physical system considered is the energy transfer in laminar, incompressible, constant properties, fully-developed flow of absorbing-emitting gases between parallel plates (Fig. 7.1). The condition of uniform surface heat flux for each plate is assumed such that the temperature of the plates varies in the axial direction. Fully developed heat transfer is considered, and axial conduction and radiation is assumed to be negligible as compared with the normal components. Consistent with the constant properties flow, the absorption coefficient is taken to be independent of temperature and radiation can be linearized. Extensive treatment of this problem is available in the literature [23, 41]. The primary motivation of studying the problem here is to investigate the extent of radiative interaction for high temperature flow conditions.

7.1 Basic Formulation

For the physical problem considered, the energy equation, Eq. (4.1), can be expressed as [8]

$$\rho C_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} + \beta T u \frac{dp}{dx} + \mu \left(\frac{\partial u}{\partial y} \right)^2 - \text{div} q_R \quad (7.1)$$

where u and v denote x and y components of velocity, respectively. In deriving Eq. (7.1) it has been assumed that the net conduction heat transfer in the x direction is negligible compared with the net conduction in the y direction. This represents the physical condition of a large value of the Peclet number. By an analogous reasoning, the radiative heat transfer in the x direction can be neglected in comparison to that transferred in the y direction. If, in addition, it is assumed that the Eckert number of the

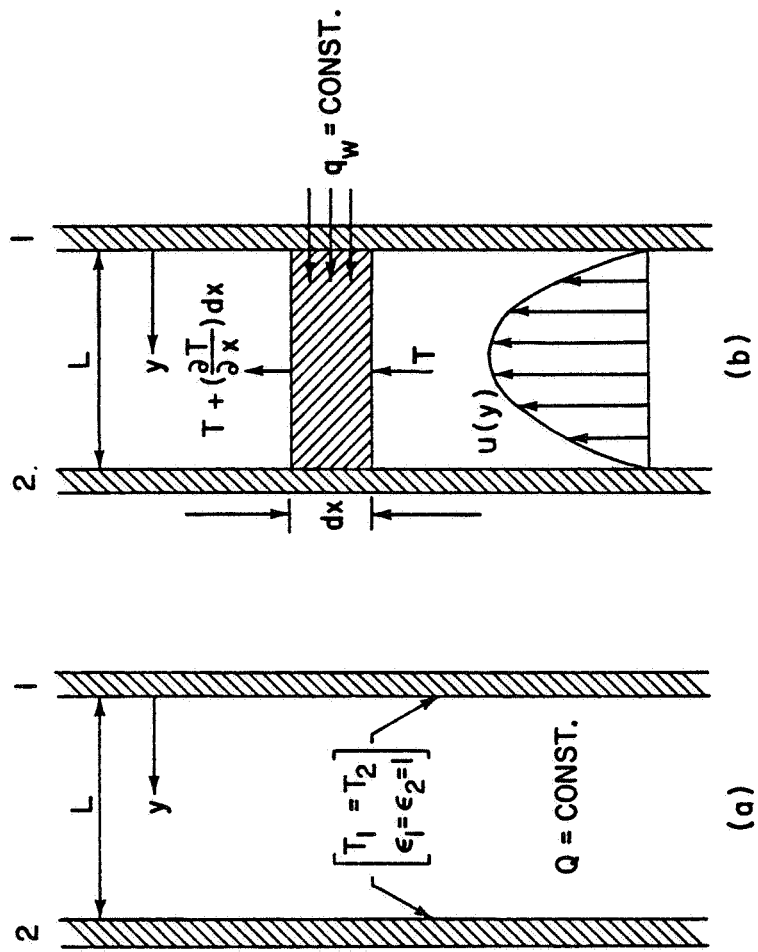


Figure 7.1 Physical model and coordinate systems for flow of radiating gases between parallel plates.

flow is small, then Eq. (7.1) reduces to [8]

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho C_p} \frac{\partial q_R}{\partial y} \quad (7.2)$$

where $\alpha = (k/\rho C_p)$ represents the thermal diffusivity of the fluid.

For a steady fully-developed flow, $v = 0$, and u is given by the well-known parabolic profile as

$$u = 6 u_m (\xi - \xi^2); \xi = y/L \quad (7.3)$$

where u_m represents the mean fluid velocity. Also, for the flow of a perfect gas with uniform wall heat flux, $\partial T/\partial x$ is constant and is given by

$$\partial T/\partial x = (2\alpha q_w)/(u_m L/k) \quad (7.4)$$

Now, by combining Eqs. (7.2) - (7.4), the energy equation is expressed in nondimensional form as

$$\frac{\partial \theta}{\partial \tau} = 12(\xi - \xi^2) = \frac{\partial^2 \theta}{\partial \xi^2} - \frac{1}{q_w} \frac{\partial q_R}{\partial \xi} \quad (7.5)$$

where

$$\tau = \alpha t/L^2; \theta = (T - T_1)/(q_w L/k)$$

The expression for $\partial q_R/\partial \xi$ in Eq. (7.5) is obtained from either Eq. (3.22) or Eq. (3.24).

By assuming that the initial temperature distribution in the gas is

some uniform value $T_0 = T_1$, the initial and boundary conditions for this problem can be expressed as

$$\theta(\xi, 0) = 0 \quad (7.6a)$$

$$\theta(0, \tau) = \theta(1, \tau) = 0 \quad (7.6b)$$

$$\theta_\xi(\xi = 1/2) = 0 ; \theta_\xi(\xi = 0) = -\theta_\xi(\xi = 1) \quad (7.6c)$$

It should be noted that all the boundary conditions given in Eqs. (7.6) are not independent and any two convenient conditions can be used to obtain solutions. Also, the initial temperature distribution can be any specified or calculated value of $\theta(\xi, 0) = f(\xi)$.

For flow problems, the quantity of primary interest is the bulk temperature of the gas, which may be expressed as [41]

$$\theta_b = (T_b - T_1)/(q_w L/k) = 6 \int_0^1 \theta(\xi, \tau) (\xi - \xi^2) d\xi \quad (7.7)$$

The heat transfer q_w is given by the expression, $q_w = h_c (T_1 - T_b)$, where h_c is the convective heat transfer coefficient (W/cm^2-K). In general, the heat transfer results are expressed in terms of the Nusselt number $Nu = h_c D_h/k$. Here, D_h represents the hydraulic diameter, and for the parallel plate geometry it equals twice the plate separation, i.e., $D_h = 2L$. Upon eliminating the convective heat transfer coefficient h_c from the expressions for q_w and Nu , a relation between the Nusselt number and the bulk temperature is obtained as

$$Nu = 2 L q_w / k(T_1 - T_b) = -2/\theta_b \quad (7.8)$$

The heat transfer results, therefore, can be expressed either in terms of Nu or θ_b .

7.1.1 Steady Laminar Flow

For steady-state conditions, $\partial\theta/\partial\tau = 0$ and Eq. (7.5) becomes

$$\theta'' - 12(\xi - \xi^2) = (1/q_w) dq_R/d\xi \quad (7.9)$$

By integrating Eq. (7.9) once and using the conditions that at $\xi = 1/2$, $q_R(\xi)$ and $(d\theta/d\xi)$ are equal to zero, one obtains

$$\theta' - 2(3\xi^2 - 2\xi^3) + 1 = q_R(\xi)/q_w \quad (7.10)$$

The expression for $q_R(\xi)$ in Eq. (7.10) is obtained from either Eq. (3.21) or Eq. (3.23).

For the present physical problem, $e_1 = e_2$ and $F_{1\omega_i} = F_{2\omega_i}$. Thus, for the case of linearized radiation, a combination of Eqs. (3.21), (5.2a), and (7.10) results in

$$\begin{aligned} \theta' - 2(3\xi^2 - 2\xi^3) + 1 &= \frac{3}{2} (L/k) \sum_{i=1}^n H_{1i} u_{0i} \left\{ \int_0^\xi \theta(\xi') \bar{A}_i' \left[\frac{3}{2} u_{01} (\xi - \xi') \right] d\xi' \right. \\ &\quad \left. - \int_\xi^1 \theta(\xi') \bar{A}_i' \left[\frac{3}{2} u_{0i} (\xi' - \xi) \right] d\xi' \right\} \end{aligned} \quad (7.11a)$$

A combination of Eqs. (3.23), (5.2a), and (7.10) gives an alternate form of the energy equation for the steady case as

$$\begin{aligned}
\theta' &= 2(3\xi^2 - 2\xi^3) + 1 \\
&= (L/k) \sum_{i=1}^n H_{1i} \left\{ \int_0^\xi (d\theta/d\xi') \bar{A}_i \left[\frac{3}{2} u_{0i}(\xi - \xi') \right] d\xi' \right. \\
&\quad \left. + \int_\xi^1 (d\theta/d\xi') \bar{A}_i \left[\frac{3}{2} u_{0i}(\xi' - \xi) \right] d\xi' \right\} \quad (7.11b)
\end{aligned}$$

Note that this equation can be obtained directly by integrating the left-hand side of Eq. (7.11a) by parts. Equations (7.11) provide two forms of the energy equation for the steady-state conditions.

For the case of negligible radiation, Eqs. (7.11) reduce to

$$\theta' = 2(3\xi^2 - 2\xi^3) - 1 \quad (7.12)$$

The solution of Eqs. (7.12) is found to be

$$\theta(\xi) = \xi(2\xi^2 - \xi^3 - 1) \quad (7.13)$$

Thus, a combination of Eqs. (7.7) and (7.13) gives the result for the bulk temperature for the steady case with no radiation as

$$-\theta_b = 17/70 \quad (7.14)$$

This result is useful in determining the extent of radiative contributions.

7.1.2 Transient Radiative Interactions

For the transient case, a combination of Eqs. (3.22), (5.2a), and (7.5) gives the energy equation for the linearized radiation as

$$\begin{aligned}
\theta_{\xi\xi} - \theta_{\tau} - 3N\theta - 12(\xi - \xi^2) \\
= \frac{9}{4} (L/k) \sum_{i=1}^n H_{1i} u_{0i}^2 \left\{ \int_0^{\xi} \theta(\xi', \tau) \bar{A}_i'' \left[\frac{3}{2} u_{0i}(\xi - \xi') \right] d\xi' \right. \\
\left. + \int_{\xi}^1 \theta(\xi', \tau) \bar{A}_i'' \left[\frac{3}{2} u_{0i}(\xi' - \xi) \right] d\xi' \right\}
\end{aligned} \tag{7.15a}$$

where

$$N = (PL^2/k) K_1 = (PL^2/k) \sum_{i=1}^n S_i(T) (de_{\omega_i}/dT)_{T_1}$$

Note that this definition of N is slightly different than the definition of N_1 in Eq. (5.4c). The dimensionless gas property N characterizes the relative importance of radiation versus conduction within the gas under optically thin conditions [23, 41]. Also, by combining Eqs. (3.24), (5.2a), and (7.5) another form of the transient energy equation is obtained as

$$\begin{aligned}
\theta_{\xi\xi} - \theta_{\tau} - 12(\xi - \xi^2) \\
= \frac{3}{2} (L/k) \sum_{i=1}^n H_{1i} u_{0i} \left\{ \int_0^{\xi} (\partial\theta/\partial\xi') \bar{A}_i' \left[\frac{3}{2} u_{0i}(\xi - \xi') \right] d\xi' \right. \\
\left. - \int_{\xi}^1 (\partial\theta/\partial\xi') \bar{A}_i' \left[\frac{3}{2} u_{0i}(\xi' - \xi) \right] d\xi' \right\}
\end{aligned} \tag{7.15b}$$

Note again that Eq. (7.15b) can be obtained directly by integrating the left-hand side of Eq. (7.15a) by parts. Quite often, Eq. (7.15b) is the convenient form to use in radiative transfer analyses.

For the case of negligible radiation, $N = 0$ and both forms of Eq. (7.15) reduce to

$$\theta_{\xi\xi} - \theta_{\tau} = 12 (\xi - \xi^2) \quad (7.16)$$

By employing the product solution procedure, the solution of Eq. (7.16) can be obtained and the result can be expressed in terms of the bulk temperature through use of Eq. (7.7).

The solution of Eq. (7.16) is assumed to be of the form

$$\theta(\xi, \tau) = g(\xi) + h(\xi, \tau) \quad (7.17)$$

From Eqs. (7.16) and (7.17), there is obtained two separate equations as

$$g'' = 12(\xi - \xi^2) \quad (7.18)$$

$$h_{\xi\xi} - h_{\tau} = 0 \quad (7.19)$$

The solution of Eq. (7.18) is obtained by direct integration as

$$g(\xi) = \xi(2\xi^2 - \xi^3 - 1) \quad (7.20)$$

This is the same result as given by Eq. (7.13) for the steady case if $g(\xi)$ is replaced by $\theta(\xi)$. The solution of Eq. (7.19) is found to be (see Sec. 7.2)

$$h(\xi, \tau) = \sum_{n=1}^{\infty} C_n \sin(n\pi\xi) \exp[-(n\pi)^2\tau] \quad (7.21a)$$

where

$$C_n = -2 \int_0^1 g(\xi) \sin(n\pi\xi) d\xi, \quad n = 1, 2, \dots \quad (7.21b)$$

Thus, the complete solution of Eq. (7.16) is given by

$$\theta(\xi, \tau) = \xi (2\xi^2 - \xi^3 - 1) + \sum_{n=1}^{\infty} C_n \sin(a\xi) \exp(-a^2\tau); \quad a = n\pi \quad (7.22)$$

The expression for C_n is obtained from Eqs. (7.20) and (7.21b) as

$$C_n = (4/a^5) [12 - 12a^2 + a^4] \cos(a) - 24, \quad n = 1, 2, \dots \quad (7.23)$$

where a is defined in Eq. (7.22). By combining Eqs. (7.7) and (7.22), the expression for the bulk temperature is obtained as

$$\theta_b = -17/70 + 6 \sum_{n=1}^{\infty} C_n [(1/a) + (4/a^3)] \exp(-a^2\tau) \quad (7.24)$$

where C_n is given by Eq. (7.23).

7.2 Optically Thin Limit

In the optically thin limit, the steady-state energy equations, Eqs. (7.11a) and (7.11b), reduce to

$$\theta' - 2(3\xi^2 - 2\xi^3) + 1 = \frac{3}{2} N \left[\int_0^\xi \theta(\xi') d\xi' - \int_0^1 \theta(\xi') d\xi' \right] \quad (7.25a)$$

$$\theta' - 2(3\xi^2 - 2\xi^3) + 1 = \frac{3}{2} N \left[\int_0^\xi (\xi - \xi') (d\theta/d\xi') d\xi' \right]$$

$$+ \int_{\xi}^1 (\xi' - \xi) (d\theta/d\xi') d\xi' \quad (7.25b)$$

The differentiation of Eqs. (7.25a) and (7.25b) yields the same energy equation for the optically thin limit as

$$\theta'' - 3N\theta = 12 (\xi - \xi^2) \quad (7.26)$$

The solution of Eq. (7.26) satisfying the boundary conditions $\theta(0, \tau) = 0$ and $\theta(1, \tau) = 0$ is found to be

$$\begin{aligned} \theta(\xi) = & (16/3N^2) [\sinh(-\sqrt{3N}/2)/\sinh(\sqrt{3N})] \cosh[\sqrt{3N}(\xi - 1/2)] \\ & + (4/N)(\xi^2 - \xi + 2/3N) \end{aligned} \quad (7.27a)$$

Alternately, the solution of Eq. (7.26) is written as

$$\begin{aligned} \theta(\xi) = & C_1 \exp(\sqrt{m} \xi) + C_2 \exp(-\sqrt{m} \xi) \\ & + (1/m^2) (24 - 12 m \xi + 12 m \xi^2) ; m = 3N \end{aligned}$$

The constants C_1 and C_2 are obtained by using the boundary conditions $\theta(0) = 0$ and $\theta'(1/2) = 0$, and the solution for $\theta(\xi)$ is found to be

$$\begin{aligned} \theta(\xi) = & (1/m^2) \{ -24/(1 + e^{-\sqrt{m}}) \} (e^{-\sqrt{m}} e^{\sqrt{m} \xi} + e^{-\sqrt{m} \xi}) \\ & + 24 - 12 m \xi + 12 m \xi^2 \} \end{aligned} \quad (7.27b)$$

Equations (7.27a) and (7.27b) should produce identical results. The expression for the bulk temperature, in this case, is obtained by combining

Eqs. (7.7) and (7.27b) as

$$\theta_b = \frac{576}{m^{7/2}} [(1-e^{-\sqrt{m}})/(1+e^{-\sqrt{m}})] - \frac{288}{m^3} + \frac{24}{m^2} - \frac{12}{5m} \quad (7.28a)$$

or

$$\theta_b = [1/(3N)^3] \{576(3N)^{-1/2} (\text{NEXP}) - 21.6N^2 + 72N - 288\} \quad (7.28b)$$

where

$$\text{NEXP} = \{1 - \exp[-(3N)^{1/2}]\} / \{1 + \exp[-(3N)^{1/2}]\}$$

In the optically thin limit, the transient energy equations, Eqs. (7.15a) and (7.15b), reduce to

$$\theta_{\xi\xi} - \theta_{\tau} - 3N\theta = 12(\xi - \xi^2) \quad (7.29a)$$

$$\begin{aligned} \theta_{\xi\xi} - \theta_{\tau} &= 12(\xi - \xi^2) \\ &= \frac{3}{2} N \left[\int_0^{\xi} (\partial\theta/\partial\xi') d\xi' - \int_{\xi}^1 (\partial\theta/\partial\xi') d\xi' \right] \end{aligned} \quad (7.29b)$$

Note that Eq. (7.29b) is identical to Eq. (7.29a).

The solution of Eq. (7.29) is assumed to be of the form

$$\theta(\xi, \tau) = g(\xi) + h(\xi, \tau) \quad (7.30)$$

Thus, Eq. (7.29) can be written as

$$h_{\xi\xi} - h_{\tau} - 3Nh = -g_{\xi\xi} + 3Ng + 12(\xi - \xi^2) \quad (7.31)$$

Consequently,

$$g'' - 3Ng = 12(\xi - \xi^2) \quad (7.32)$$

and

$$h_{\xi\xi} - h_{\tau} - 3Nh = 0 \quad (7.33)$$

The conditions for Eqs. (7.32) and (7.33) are obtained from Eq. (7.6) as

$$\theta(0,\tau) = h(0,\tau) + g(0) = 0; \quad h(0,\tau) = 0, \quad g(0) = 0 \quad (7.34a)$$

$$\theta(1,\tau) = h(1,\tau) + g(1) = 0; \quad h(1,\tau) = 0, \quad g(1) = 0 \quad (7.34b)$$

$$\theta(\xi,0) = h(\xi,0) + g(\xi) = 0; \quad h(\xi,0) = -g(\xi) \quad (7.34c)$$

The solution of Eq. (7.32) satisfying the boundary conditions given by Eqs. (7.34a) and (7.34b) is identical to the solution of Eq. (7.26) as given by Eq. (7.27) if $\theta(\xi)$ is replaced by $g(\xi)$, i.e.,

$$g(\xi) = (16/3N^2) [\sinh(-\sqrt{3N}/2) / \sinh(\sqrt{3N})] \cosh[\sqrt{3N}(\xi-1/2)] \\ + (4/N)(\xi^2 - \xi + 2/3N) \quad (7.35)$$

The solution of Eq. (7.33) is obtained by using the product solution procedure and implying the conditions $h(0,\tau) = h(1,\tau) = 0$ and $h(\xi,0) = -g(\xi)$. For the product solution, it is assumed that

$$h(\xi,\tau) = F(\xi) G(\tau) \quad (7.36)$$

By using Eq. (7.36), Eq. (7.33) is separated into two ordinary differential equations which are expressed along with appropriate conditions as

$$F'' + \lambda^2 F = 0; \quad F(0) = 0, \quad F(1) = 0 \quad (7.37)$$

$$\dot{G} + (3N + \lambda^2) G = 0 ; h(\xi, 0) = G(0) = g(\xi) \quad (7.38)$$

The solution of Eq. (7.37) is given by

$$F_n(\xi) = \sin(n\pi\xi), n = 1, 2, \dots$$

and the solution of Eq. (7.38) is found to be

$$G_n(\tau) = C_n \exp \{ - [3N + (n\pi)^2] \tau \}$$

Thus, the complete solution of Eq. (7.33) is

$$h(\xi, \tau) = \sum_{n=1}^{\infty} C_n \sin(n\pi\xi) \exp \{ - [3N + (n\pi)^2] \tau \} \quad (7.39)$$

where

$$C_n = -2 \int_0^1 g(\xi) \sin(n\pi\xi) d\xi, n = 1, 2, \dots \quad (7.40)$$

Now, the solution of Eq. (7.29), as expressed by Eq. (7.30), is written as

$$\begin{aligned} \theta(\xi, \tau) = & (16/3N^2) [\sinh(-\sqrt{3N}/2) / \sinh(\sqrt{3N})] \cosh[\sqrt{3N}(\xi - 1/2)] \\ & + (4/N)(\xi^2 - \xi + 2/3N) \\ & + \sum_{n=1}^{\infty} C_n \sin(n\pi\xi) \exp\{-[3N + (n\pi)^2]\tau\} \end{aligned} \quad (7.41)$$

From Eqs. (7.35) and (7.40), it follows that

$$C_n = 0, \text{ for } n \text{ even} \quad (7.42a)$$

$$= 32[3N + (n\pi)^2] / [3N^2(n\pi)^3] + 2(n\pi) / \{3N^2[3N + (n\pi)^2]\}, \text{ for } n \text{ odd} \quad (7.42b)$$

By combining Eqs. (7.7) and (7.41), the expression for the bulk temperature is obtained as

$$\begin{aligned} \theta_b = & 6 \{ (16/3N^2) [\sinh(-\sqrt{3N}/2) / \sinh(\sqrt{3N})] [(1/3N) \cosh(\sqrt{3N}/2) \\ & - (4 + \sqrt{3N}) (3N)^{-3/2} \sinh(\sqrt{3N}/2)] + (4/N) [-1/30 + 1/(9N)] \\ & + \sum_{n=1}^{\infty} C_n [1/(n\pi) + 4/(n\pi)^3] \exp[-(3N + n^2\pi^2)\tau] \} \end{aligned} \quad (7.43)$$

where C_n is given by Eq. (7.42).

7.3 Large Path Length Limit

In the large path length limit, the steady-state energy equations, Eqs. (7.11a) and (7.11b), reduce to

$$\theta' - 2(3\xi^2 - 2\xi^3) + 1 = M \int_0^1 \theta(\xi') d\xi' / (\xi - \xi') \quad (7.44a)$$

$$\begin{aligned} \theta' - 2(3\xi^2 - 2\xi^3) + 1 &= (L/k) \sum_{i=1}^n H_{1i} \left\{ \int_0^\xi (d\theta/d\xi') \ln \left[\frac{3}{2} u_{0i}(\xi - \xi') \right] d\xi' \right. \\ &\quad \left. + \int_\xi^1 (d\theta/d\xi') \ln \left[\frac{3}{2} u_{0i}(\xi' - \xi) \right] d\xi' \right\} \end{aligned} \quad (7.44b)$$

where

$$M = H_1 L/k = (L/k) \sum_{i=1}^n A_{0i} (de_{wi}/dT)_{T_1} \quad (7.44c)$$

Through integration by parts, it can be shown that Eq. (7.44b) reduces to Eq. (7.44a). The parameter M in Eq. (7.44c) is defined differently than M_1

in Eq. (5.4e). The nondimensional parameter M constitutes the radiation-conduction interaction parameter for the large path length limit [23, 41]. Equation (7.44a) does not appear to possess a closed form solution; a numerical solution, however, can be obtained easily.

In the large path length limit, the transient energy equations, Eqs. (7.15a) and (7.15b), reduce to

$$\begin{aligned} \theta_{\xi\xi} - \theta_{\tau} - 3N\theta - 12(\xi - \xi^2) \\ = - (H_1 L/k) \left[\int_0^{\xi} \theta(\xi', \tau) d\xi' / (\xi - \xi')^2 + \int_{\xi}^1 \theta(\xi', \tau) d\xi' / (\xi' - \xi)^2 \right] \end{aligned} \quad (7.45a)$$

$$\theta_{\xi\xi} - \theta_{\tau} - 12(\xi - \xi^2) = (H_1 L/k) \int_0^1 (\partial\theta/\partial\xi') d\xi' / (\xi - \xi') \quad (7.45b)$$

Since $|(\xi - \xi')^2| = |(\xi' - \xi)^2|$, Eq. (7.45a) can be written as

$$\begin{aligned} \theta_{\xi\xi} - \theta_{\tau} - 3N\theta - 12(\xi - \xi^2) \\ = - (H_1 L/k) \int_0^1 \theta(\xi', \tau) d\xi' / (\xi - \xi')^2 \end{aligned} \quad (7.45c)$$

Through integration by parts, Eq. (7.45c) can be expressed as

$$\theta_{\xi\xi} - \theta_{\tau} - 3N\theta - 12(\xi - \xi^2) = (H_1 L/k) \int_0^1 (\partial\theta/\partial\xi') d\xi' / (\xi - \xi') \quad (7.45d)$$

Equations (7.45a) - (7.45d) represent different forms of the governing equations in the large path length limit. With the exception of the term $(-3N\theta)$ on the left-hand side, Eq. (7.45d) is identical to Eq. (7.45b). Since N represents the radiation-conduction interaction parameter only in the optically thin limit [23], it should not appear in the governing equation

for the large path length limit. Thus, Eq. (7.45b) is the correct equation to use for solution in the large path length limit; the solution of this equation is obtained by numerical techniques.

7.4 Method of Solution

The solution procedures for both steady and unsteady cases are presented in this section. In principle, the same numerical procedure applies to both the general and large path length limit cases.

7.4.1 Steady-State Solutions

The general solution of Eq. (7.11a) or Eq. (7.11b) is obtained numerically by employing the method of variation of parameters. For this, a polynomial form for $\theta(\xi)$ is assumed in powers of ξ as

$$\theta(\xi) = \sum_{m=0}^n a_m \xi^m \quad (7.46)$$

By considering a five term series solution (a quartic solution in ξ) and satisfying the boundary conditions $\theta(0) = \theta'(1/2) = 0$ and $\theta'(0) = -\theta'(1)$, one obtains

$$\theta(\xi) = a_1(\xi - 2\xi^3 + \xi^4) + a_2(\xi^2 - 2\xi^3 + \xi^4) \quad (7.47)$$

Thus,

$$\theta'(\xi) = a_1(1 - 6\xi^2 + 4\xi^3) + a_2(2\xi - 6\xi^2 + 4\xi^3) \quad (7.48)$$

A substitution of Eq. (7.48) in Eq. (7.11a) results in

$$a_1(1 - 6\xi^2 + 4\xi^3) + a_2(2\xi - 6\xi^2 + 4\xi^3) - 2(3\xi^2 - 2\xi^3) + 1$$

$$\begin{aligned}
&= \frac{3}{2} (L/k) \sum_{i=1}^n H_{1i} u_{0i} \left\{ \int_0^\xi \theta(\xi') \bar{A}_i' \left[\frac{3}{2} u_{0i} (\xi - \xi') \right] d\xi' \right. \\
&\quad \left. - \int_\xi^1 \theta(\xi') \bar{A}_i' \left[\frac{3}{2} u_{0i} (\xi' - \xi) \right] d\xi' \right\} \quad (7.49)
\end{aligned}$$

where expressions for $\theta(\xi')$ are obtained from Eq. (7.47).

The two unknown constants a_1 and a_2 in Eq. (7.49) are evaluated by satisfying the integral equation at two convenient locations ($\xi=0$ and $\xi = 1/4$ in the present case). The entire procedure for obtaining a_1 and a_2 is described in Appendix E from which it follows that

$$a_1 = (11\alpha_2 - 16\alpha_4)/\text{DEN} \quad (7.50a)$$

$$a_2 = (16\alpha_3 - 11\alpha_1)/\text{DEN} \quad (7.50b)$$

where

$$\text{DEN} = 16 (\alpha_1\alpha_4 - \alpha_2\alpha_3) \quad (7.50c)$$

and coefficients α_1 through α_4 are defined in Appendix E.

Now, with known values of a_1 and a_2 , Eq. (7.47) provides the general solution for $\theta(\xi)$. The expression for the bulk temperature is obtained by combining Eqs. (7.7) and (7.47) as

$$\theta_b = (1/70) (17a_1 + 3a_2) \quad (7.51)$$

Note that for the case of no radiative interaction α_2 , α_3 , and α_4 are equal to zero and $\alpha_1 = 1$. Thus, $a_2 = 0$ and $a = -1$, and Eq. (7.51) gives the result of Eq. (7.14).

The governing equation for the large path length limit is Eq. (7.44a).

For this equation also the solution is given by Eq. (7.47) but the values of α 's are completely different in this case. There are two approaches to obtain solutions in the large path length limit. One approach is to make use of Eq. (7.44a) and go through the entire numerical procedure described in Appendix E for the general solution. Another approach is to work with the general solution but evaluate all R_i and S_i integrals of Appendix E in the large path length limit. In the large path length limit, the integrals can be evaluated in closed forms. Both procedures are described briefly in Appendix E. In order to distinguish the large path length limit solution from the general solution, constants a_1 and a_2 are replaced by b_1 and b_2 , and coefficients α_1 through α_4 are replaced by β_1 through β_4 . The solution for the large path length limit, therefore, is given by

$$\theta(\xi) = b_1(\xi - 2\xi^3 + \xi^4) + b_2(\xi^2 - 2\xi^3 + \xi^4) \quad (7.52a)$$

where

$$b_1 = (11\beta_2 - 16\beta_4)/\text{BOTTOM} \quad (7.52b)$$

$$b_2 = (16\beta_3 - 11\beta_1)/\text{BOTTOM} \quad (7.52c)$$

$$\text{BOTTOM} = 16(\beta_1\beta_4 - \beta_2\beta_3) \quad (7.52d)$$

and coefficients β_1 through β_4 are defined in Appendix E. For this case, the bulk temperature is given by

$$\theta_b = (1/70)(17b_1 + 3b_2) \quad (7.53)$$

Note that in this case the value of coefficients β_1 through β_4 are obtained in closed form.

7.4.2 Transient Solutions

The governing energy equations for the transient case are Eqs. (7.15a) and (7.15b). As in Sec. 6.3, general solutions of these equations are obtained numerically by employing the method of variation of parameters. For the present problem, a polynomial form for $\theta(\xi, \tau)$ is assumed as

$$\theta(\xi, \tau) = \sum_{m=0}^n a_m(\tau) \xi^m \quad (7.54)$$

For a quadratic temperature distribution in ξ (with time dependent coefficients), Eq. (7.54) is written as

$$\theta(\xi, \tau) = a_0(\tau) + a_1(\tau) \xi + a_2(\tau) \xi^2 \quad (7.55a)$$

By using the boundary conditions $\theta(0, \tau) = 0$ and $\theta_{\xi}(\xi=1/2) = 0$, this reduces to

$$\theta(\xi, \tau) = g(\tau) (\xi - \xi^2) \quad (7.55b)$$

where $g(\tau)$ represents the time dependent coefficient. Consequently,

$$\theta_{\xi}(\xi, \tau) = g(\tau) (1 - 2\xi); \theta_{\xi\xi}(\xi, \tau) = -2g(\tau); \theta_{\tau}(\xi, \tau) = (\xi - \xi^2)g'(\tau) \quad (7.56)$$

Also, a combination of Eq. (7.6a) and (7.55b) yields the initial condition

$$\theta(\xi, 0) = g(0) = 0 \quad (7.57)$$

Note that essential boundary conditions are used already in obtaining the

solution represented by Eq. (7.55b).

By employing Eqs. (7.55b) and (7.56), Eqs. (7.15a) and (7.15b) are transformed in alternate forms which are expressed in a compact form as

$$g'(\tau) + \begin{bmatrix} J_1(\xi) \\ J_2(\xi) \end{bmatrix} g(\tau) + 12 = 0 \quad (7.58)$$

where $J_1(\xi)$ and $J_2(\xi)$ are defined in Appendix F. The function $J_1(\xi)$ is used for solution of Eq. (7.15a) and $J_2(\xi)$ is used for solution of Eq. (7.15b). The solution of Eq. (7.58) satisfying the initial conditions of Eq. (7.57) is given by

$$g(\tau) = \frac{12}{J(\xi)} \{ \exp [-J(\xi)\tau] - 1 \} \quad (7.59)$$

The temperature distribution given by Eq. (7.56b) can be expressed now as

$$\theta(\xi, \tau) = \frac{12}{J(\xi)} \{ \exp [-J(\xi)\tau] - 1 \} (\xi - \xi^2) \quad (7.60)$$

The expression for the bulk temperature is obtained through use of Eq. (7.7) as

$$\theta_b = 72 \int_0^1 [(\xi - \xi^2)^2 / J(\xi)] \{ \exp [-J(\xi)\tau] - 1 \} \quad (7.61)$$

Note that in Eqs. (7.59)-(7.61), $J(\xi)$ becomes $J_1(\xi)$ for solution of Eq. (15a) and $J_2(\xi)$ for solution of Eq. (15b).

For a quartic solution in ξ , Eq. (7.54) gives the result identical to

Eq. (7.47) which for the transient case is expressed as

$$\theta(\xi, \tau) = g(\tau)(\xi - 2\xi^3 + \xi^4) + h(\tau)(\xi^2 - 2\xi^3 + \xi^4) \quad (7.62)$$

Thus,

$$\theta_{\xi}(\xi, \tau) = (1 - 6\xi^2 + 4\xi^3)g(\tau) + 2(\xi - 3\xi^2 + 2\xi^3)h(\tau) \quad (7.63a)$$

$$\theta_{\xi\xi}(\xi, \tau) = 12(-\xi + \xi^2)g(\tau) + 2(1 - 6\xi + 6\xi^2)h(\tau) \quad (7.63b)$$

$$\theta_{\tau}(\xi, \tau) = (\xi - 2\xi^3 + \xi^4)g'(\tau) + (\xi^2 - 2\xi^3 + \xi^4)h'(\tau) \quad (7.63c)$$

By substituting Eqs. (7.62) and (7.63) into Eq. (7.15a), one obtains

$$x g'(\tau) + J_3(\xi) g(\tau) + y h' + J_4(\xi) h(\tau) = -z \quad (7.64)$$

where

$$x = (\xi - 2\xi^3 + \xi^4); y = (\xi^2 - 2\xi^3 + \xi^4); z = 12(\xi - \xi^2)$$

and functions $J_3(\xi)$ and $J_4(\xi)$ are defined in Appendix F. Equation (7.64) constitutes one equation in two unknowns, namely $g(\tau)$ and $h(\tau)$. However, since the equation is linear in τ , the principle of superposition can be used to split the solution into two solutions as

$$x g'(\tau) + J_3(\xi) g(\tau) = -z/2 \quad (7.65)$$

$$y h'(\tau) + J_4(\xi) h(\tau) = -z/2 \quad (7.66)$$

The initial condition for this case can be written as

$$\theta(\xi, 0) = g(0)(\xi - 2\xi + \xi^3) + h(0)(\xi^2 - 2\xi^3 + \xi^4) = 0 \quad (7.67a)$$

Consequently,

$$g(0) = 0; h(0) = 0 \quad (7.67b)$$

The solution of Eqs. (7.65) and (7.66) satisfying the appropriate initial condition of Eq. (7.67b) is given respectively as

$$g(\tau) = [z(\xi)/2J_3(\xi)] \{ \exp[-J_3(\xi)\tau/x(\xi)] - 1 \} \quad (7.68)$$

$$h(\tau) = [z(\xi)/2J_4(\xi)] \{ \exp[-J_4(\xi)\tau/y(\xi)] - 1 \} \quad (7.69)$$

By substituting Eqs. (7.68) and (7.69) into Eq. (7.62), the expression for the temperature distribution is obtained as

$$\begin{aligned} \theta(\xi, \tau) = & [6(\xi - \xi^2)(\xi - 2\xi^3 + \xi^4)/J_3(\xi)] \{ \exp[-J_3(\xi)\tau/x(\xi)] - 1 \} \\ & + [6(\xi - \xi^2)(\xi^2 - 2\xi^3 + \xi^4)/J_4(\xi)] \{ \exp[-J_4(\xi)\tau/y(\xi)] - 1 \} \end{aligned} \quad (7.70)$$

The bulk temperature in this case is given by

$$\begin{aligned} \theta_b = & 36 \int_0^1 [(\xi - \xi^2)(\xi - 2\xi^3 + \xi^4)/J_3(\xi)] \{ \exp[-J_3(\xi)\tau/x(\xi)] - 1 \} d\xi \\ & + 36 \int_0^1 [(\xi - \xi^2)(\xi^2 - 2\xi^3 + \xi^4)/J_4(\xi)] \{ \exp[-J_4(\xi)\tau/y(\xi)] - 1 \} d\xi \end{aligned} \quad (7.71)$$

where x and y are defined in Eq. (7.64).

By substituting Eqs. (7.62) and (7.63) into Eq. (7.15b), there is obtained

$$xg' + J_5(\xi) g(\tau) + yh' + J_6(\xi) h(\tau) = -z \quad (7.72)$$

where again x, y, z are defined in Eq. (7.64) and functions $J_5(\xi)$ and $J_6(\xi)$ are defined in Appendix F. The solution procedure for this equation is

identical to that for Eq. (7.64) and the results for temperature distribution and bulk temperature are given respectively by Eqs. (7.70) and (7.71) with J_3 replaced by J_5 and J_4 by J_6 .

In the large path length limit, the two applicable governing equations are Eqs. (7.45b) and (7.45d). The solutions of these equations can be obtained from the general solutions by evaluating the integrals in J function in the large path length limit.

Alternately, for a quadratic temperature distribution, Eqs. (7.45d) and (7.45b) are transformed respectively to

$$g'(\tau) + \left[\frac{J_7(\xi)}{J_8(\xi)} \right] g(\tau) + 12 = 0 \quad (7.73)$$

where $J_7(\xi)$ and $J_8(\xi)$ are defined in Appendix F. The solution of Eq. (7.73) is given by Eq. (7.59) and expressions for $\theta(\xi, \tau)$ and θ_b can be obtained from Eqs. (7.60) and (7.61) respectively. Of course, proper care should be taken to use the correct relation for J functions for different equations. The large path length limit solutions for a quartic temperature distribution can be obtained in a similar manner.

8. PLANS FOR SPECIFIC RESULTS

Some specific results have been obtained and these are being analyzed. The present plans are to obtain extensive results for the following cases with varying physical and flow conditions:

A. Physical Geometries

1. Parallel Plates: One-Dimensional Radiation
2. Parallel Plates: Two-Dimensional Radiation
3. Diffusing Channel Flow: One- and Two-Dimensional Radiation
4. Channel Flow: Top Plate Flat, Bottom Plate with a 5-15° ramp
5. Scramjet Inlet Configurations

B. Radiative Interaction Cases

1. Transient Radiative Transfer in Homogeneous Gases
2. Transient Energy Transfer By Radiation and Conduction in Homogeneous Gaseous Systems
3. Transient Energy Transfer By Radiation, Conduction, and Convection in Homogeneous Gaseous Systems
4. Applications to Flow of Homogeneous Gaseous Mixture
5. Applications to Flow of Chemically Reacting Gaseous Mixtures
6. Applications to the Scramjet Inlet Configurations.

C. Boundary Conditions

1. Isothermal Black Boundaries
2. Isothermal Gray Boundaries
3. Nonisothermal Boundaries
4. Boundaries with Uniform Heat Flux
5. Actual Scramjet Inlet Configurations

D. Flow Conditions

1. Incompressible - Various Cases
2. Compressible - $M_\infty = 0.5, 1, 2, 3, 4, \text{ and } 5$
3. Temperature Range - 300, 500, 1000, 2000, and 5000 K
4. Pressure Range - 0.1, 1, 2, 5, and 10 atm.
5. Realistic Conditions for Scramjet Inlet Flows

E. Participating Medium

1. CO - One Fundamental Band
2. CO₂ - Three Important Bands
3. H₂O - Five Important Bands
4. OH - One Fundamental Band
5. CO₂ + H₂O (Different Concentrations)
6. OH + H₂O (Different Concentrations)
7. OH + H₂ + O₂ + H₂O (Different Concentrations)
8. A Realistic Combustion Model for the Hydrogen Air System

F. Specific Results

1. Nongray Solutions Based on Band-Model Correlations
2. Optically Thin Solutions
3. Large Path Length Limit Solutions
4. Gray Solutions
5. Nongray Solutions for Scramjet Inlet Flows

9. CONCLUDING REMARKS

A brief review is presented on various band models and band model correlations that are useful in nongray radiative transfer analyses. Different formulations for one-dimensional radiative flux are provided. These are used to develop the basic governing equations for transient energy transfer in gaseous systems. Limiting forms of these equations are obtained in the optically thin and large path length limits. Numerical procedures are described to solve the governing equations for different physical and flow conditions. The plans for obtaining extensive results for different cases are provided. The formulation and numerical procedure presented in this study can be extended easily to multi-dimensional analyses. In the near future, the influence of radiative interactions will be investigated for the realistic flow of hydrogen-air mixture in the scramjet inlet configuration.

REFERENCES

1. Chandrasekhar, S., Radiative Transfer, Dover Publications, Inc., New York, 1960.
2. Kourganoff, V., Basic Methods in Transfer Problems, Dover Publications, Inc., New York, 1963.
3. Sobolev, V. V., A Treatise on Radiative Transfer, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1963.
4. Goody, R.M., Atmospheric Radiation I: Theoretical Basis, Oxford University Press, London and New York, 1965.
5. Preisendorfer, R. W., Radiative Transfer on Discrete Spaces, Pergamon Press, London and New York, 1965.
6. Vincenti, W. G. and Kruger, C. H., Introduction to Physical Gas Dynamics, John Wiley & sons, Inc., New York, 1965. Printed and Published by Robert E. Krieger Publishing Co., Inc., Huntington, N.Y.
7. Pai, Shih-I., Radiation Gas Dynamics, Springer-Verlag New York Inc., New York, 1966.
8. Sparrow, E. M. and Cess, R. D., Radiation Heat Transfer, Brooks/Cole, Belmont, Calif., 1966 and 1970. New Augmented Edition, Hemisphere Publishing Corp., Washington, D.C., 1978.
9. Hottel, H.C. and Sarofim, A. F., Radiative Transfer, McGraw-Hill Book Co., New York, 1967.
10. Love, T. J., Radiative Heat Transfer, Charles E. Merrill Publishing Co., Columbus, Ohio, 1968.
11. Penner, S. S. and Olfe, D. B., Radiation and Reentry, Academic Press, New York, 1968.
12. Kondratyev, K. Ya., Radiation in Atmosphere, Academic Press, New York, 1969.
13. Siegel, R. and Howell, J. R., Thermal Radiation Heat Transfer, McGraw-Hill Book Co., New York, 1971; Second Edition, 1981.
14. Ozisik, M. N., Radiative Transfer and Interactions with Conduction and Convection, John Wiley & Sons, Inc., 1973.
15. Pomraning, G. C., Radiation Hydrodynamics, Pergamon Press, London and New York, 1973.
16. Paltridge, G. and Platt, C., Radiation Process in Meteorology and Climatology, American Elsevier Publishing Co., Inc., New York, 1976.
17. Rybicki, G. B. and Lightman, A. P., Radiative Processes in Astrophysics, John Wiley and Sons, New York, 1979.

18. Liou, K. N., An Introduction to Atmospheric Radiation, Academic Press, 1980.
19. Cess, R. D., "The Interaction of Thermal Radiation with Conduction and Convection Heat Transfer," Advances in Heat Transfer, Vol. 1, Academic Press, New York, 1964.
20. Sparrow, E. M., "Radiation Heat Transfer between Surfaces," Advances in Heat Transfer, Vol. 2, Academic Press, New York, 1965.
21. Viskanta, R., "Radiative Transfer and Interaction of Convection with Radiation Heat Transfer," Advances in Heat Transfer, Vol. 3, Academic Press, New York, 1966.
22. Tien, C. L., "Thermal Radiation Properties of Gases," Advances in Heat Transfer, Vol. 5, Academic Press, New York, 1968.
23. Cess, R. D. and Tiwari, S. N., "Infrared Radiative Energy Transfer in Gases," Advances in Heat Transfer, Vol. 8, Academic Press, New York, 1972.
24. Edwards, D. K., "Molecular Gas Band Radiation," Advances in Heat Transfer, Vol. 12, Academic Press, New York, 1976.
25. Tiwari, S. N., "Band Models and Correlations for Infrared Radiation," Radiative Transfer and Thermal Control (Progress in Astronautics and Aeronautics), Vol. 49, American Institute of Aeronautics and Astronautics, New York, 1976.
26. Tiwari, S. N., "Models for Atmospheric Radiation," Advances in Geophysics, Vol. 20, Academic Press, New York, 1978.
27. McClatchey, R. A., Benedict, W. S., Clough, S. A., Burch, D. E., Calfee, R. F., Fox, K., Rothman, L. S., and Garing, J. S., "AFCRL Atmospheric Line Parameters Compilation," Air Force Cambridge Research Laboratories, Bedford, Mass., AFCRL-TR-73-0096, January 1973 (also, NTIS AD A762904).
28. Ludwig, C. B., Malkmus, W., Reardon, J. E., and Thomson, J. A. L., Handbook of Infrared Radiation from Combustion Gases, R. Goulard and J. A. L. Thomson (Editors), NASA SP-3080, 1973.
29. Ludwig, C.B. (Editor), Atmospheric Effects on Radiative Transfer, Proceedings of the Society of Photo-Optical Instrumentation Engineers, Vol. 195, August 1979.
30. Rothman, L. S., Gamache, R. R., Barbe, A., Goldman, A., Gillis, J. R., Brown, L. R., Toth, R. A., Flaud, J. M., and Camy-Peyret, C., "AFGL Atmospheric Absorption Line Parameters Compilation: 1982 Edition," Applied Optics, Vol. 22, No. 15, August 1983, pp. 2247-2256.
31. Plass, G. N., "Models for Spectral Band Absorption," Journal of the Optical Society of America, Vol. 8, No. 10, October 1958, pp. 690-703.

32. Plass, G. N., "Useful Representation for Measurements of Spectral Band Absorption," Journal of the Optical Society of America, Vol. 50, No. 9 September 1960, pp. 868-875.
33. Wyatt, P. J., Stull, V. R., and Plass, G. N., "Quasi-Random Model of Band Absorption," Journal of the Optical Society of America, Vol. 52, No. 11, November 1962, pp. 1209-1217.
34. Edwards, D. K. and Menard, W. A., "Comparison of Methods for Correlation of Total Band Absorption," Applied Optics, Vol. 3, No. 5, May 1964, pp. 621-625.
35. Edwards, D. K., Glassen, L. K., Hauser, W. C., and Tucher, J.S., "Radiation Heat Transfer in Nonisothermal Nongray Gases," Journal of Heat Transfer, Vol. 89C, No. 3, 1967, pp., 219-229.
36. Edwards, D. K. and Balakrishnan, A., "Slab Band Absorptance for Molecular Gas Radiation," Journal of the Quantitative Spectroscopy and Radiative Transfer, Vol. 12, 1972, pp. 1379-1387.
37. Edwards, D. K. and Balakrishnan, A., "Thermal Radiation by Combustion Gases," International Journal of Heat and Mass Transfer, Vol. 16, No. 1, January 1973, pp. 25-40.
38. Felske, J. D. and Tien, C. L., "A Theoretical Closed Form Expression for the Total Band Absorptance of Infrared Radiating Gases," International Journal of Heat and Mass Transfer, Vol. 17, No. 1, January 1974, pp., 155-158.
39. Goody, R. M. and Belton, M. J. S., "Radiative Relaxation Times for Mars (A Discussion of Martian Atmospheric Dynamics)," Planetary and Space Science, Vol. 1r, No. 2, 1967, pp. 247-256.
40. Tien, C. L. and Ling, G. R., "On a Simple Correlation for Total Band Absorptance of Radiating Gases," International Journal of Heat and Mass Transfer, Vol. 12, No. 9, September 1969, pp. 1179-1181.
41. Tiwari, S. N., "Applications of Infrared Band Model Correlations to Nongray Radiation," International Journal of Heat and Mass Transfer, Vol. 20, No. 7, July 1977, pp. 741-751.
42. Lick, W., "Transient Energy Transfer by Radiation and Conduction," International Journal of Heat and Mass Transfer, Vol. 8, 1965, pp. 119-127.
43. Doornink, D. G. and Hering, R. G., "Transient Radiative Heat Transfer in a Nongray Medium," Journal of the Quantitative Spectroscopy and Radiative Transfer, Vol. 12, 1972, pp. 1161-1174.
44. Larson, D. W. and Viskanta, R., "Transient Combined Laminar Free Convection and Radiation in a Rectangular Enclosure," Journal of Fluid Mechanics, Vol. 78, Part 1, 1976, pp. 65-85.
45. Melnikov, V. I. and Sukhovich, E. P., "Transient Heat Exchange Between a Radiating Plate and a High-Temperature Gas Flow," Heat Transfer-Soviet Research, Vol. 10, No. 3, May-June 1978, pp. 11-20 (Translation).

APPENDICES

APPENDIX A

EXPONENTIAL INTEGRALS AND EXPRESSIONS FOR RADIATIVE FLUX

Some important relations for the exponential integrals are given in Appendix B of Ref. 8; and it is noted that

$$\int E_n(t) dt = -E_{n+1}(t); \quad E_n(0) = 1/(n-1) \quad . \quad (A.1)$$

Now, consider the first integral in Eq. (3.1) as

$$I(1) = \int_0^{\tau_\lambda} E_2(\tau_\lambda - t) dt \quad (A.2)$$

By defining $x = \tau_\lambda - t$, $dt = -dx$, and Eq. (A.2) becomes

$$\begin{aligned} I(1) &= \int_{\tau_\lambda}^0 E_2(x)(-dx) = \int_0^{\tau_\lambda} E_2(x) dx \\ &= -[E_3(x)]_0^{\tau_\lambda} = E_3(0) - E_3(\tau_\lambda) \quad . \end{aligned}$$

Thus,

$$E_3(\tau_\lambda) = E_3(0) - I(1) = \frac{1}{2} - \int_0^{\tau_\lambda} E_2(\tau_\lambda - t) dt \quad (A.3)$$

The second integral in Eq. (3.2) is written as

$$I(2) = \int_{\tau_\lambda}^{\tau_{0\lambda}} E_2(t - \tau_2) dt \quad . \quad (A.4)$$

By defining $x = t - \tau_\lambda$, Eq. (A.4) is expressed as

$$\begin{aligned} I(2) &= \int_0^{\tau_{0\lambda} - \tau_\lambda} E_2(x) dx = [E_3(x)]_0^{\tau_{0\lambda} - \tau_\lambda} \\ &= - [E_3(\tau_{0\lambda} - t_\lambda) - E_3(0)] \end{aligned}$$

or

$$E_3(\tau_{0\lambda} - \tau_\lambda) = E_3(0) - I(2) = \frac{1}{2} - \int_{\tau_\lambda}^{\tau_{0\lambda}} E_2(t - \tau_\lambda) dt \quad . \quad (A.5)$$

A substitution of Eqs. (A.3) and (A.5) into Eq. (3.1) results in (for the case $B_{1\lambda} = e_{1\lambda}$ and $B_{2\lambda} = e_{2\lambda}$)

$$\begin{aligned} q_{R\lambda} &= e_{1\lambda} - e_{2\lambda} \\ &\quad + 2 \left\{ \int_0^{\tau_\lambda} [e_{b\lambda}(t) - e_{1\lambda}] E_2(\tau_\lambda - t) dt \right\} \\ &\quad - 2 \left\{ \int_{\tau_\lambda}^{\tau_{0\lambda}} [e_{b\lambda}(t) - e_{2\lambda}] E_2(t - \tau_\lambda) dt \right\} \end{aligned} \quad (A.6)$$

This equation when expressed in terms of the wave number ω is exactly the same as Eq. (3.4). Following a similar procedure, Eq. (3.5) is obtained from Eq. (3.2).

APPENDIX B

ALTERNATE FORMS OF RADIATIVE FLUX EQUATIONS

In radiative formulations, it is desirable to express the relations for q_r and $\text{div } q_r$ in terms of \bar{A} and \bar{A}' , and avoid the use of \bar{A}'' . This is accomplished by expressing the integrals containing \bar{A}' and \bar{A}'' in alternate forms through the procedure of integration by parts. This is performed by using the relation

$$\int_a^b m \, dn = (mn)_a^b - \int_a^b n \, dm \quad (\text{B.1})$$

Consider now the first integral in Eq. (3.21) and express as

$$\text{BI}(1) = \int_0^\xi F_{1\omega}(\xi') \bar{A}' \left[\frac{3}{2} u_{0i}(\xi - \xi') \right] d\xi' \quad (\text{B.2})$$

For integration by parts, let

$$m = F_{1\omega}; \quad dn = \bar{A}' \left[\frac{3}{2} u_0(\xi - \xi') \right] d\xi'.$$

Then,

$$dm = (dF_{1\omega}/d\xi') d\xi' = (de/d\xi') d\xi'.$$

In order to get n , let

$$u = \frac{3}{2} u_0(\xi - \xi'), \quad du/d\xi' = -3 u_0/2$$

and

$$\begin{aligned} dn &= [d\bar{A}(u)/du] d\xi' = [d\bar{A}(u)/d\xi' \times d\xi'/du] d\xi' \\ &= - (2/3u_0) [d\bar{A}(u)/d\xi'] d\xi' = (-2/3u_0) d\bar{A}(u). \end{aligned}$$

Thus,

$$n = -(2/3u_0) \bar{A}(u) = -(2/3u_0) \bar{A} \left[\frac{3}{2} u_0 (\xi - \xi') \right] .$$

Consequently Eq. (B.2) can be written as

$$\begin{aligned} BI(1) = & F_{1\omega}(\xi) \left[-\frac{2}{3u_0} \bar{A}(0) \right] - F_{1\omega}(0) \left[-\frac{2}{3u_0} \bar{A} \left(\frac{3}{2} u_0 \xi \right) \right] \\ & + \frac{2}{3u_0} \int_0^\xi [de_\omega(\xi')/d\xi'] \bar{A} \left[\frac{3}{2} u_0 (\xi - \xi') \right] d\xi' \end{aligned} \quad (B.3)$$

Note that by the definitions given in Eqs. (2.2) and (2.8), $\bar{A}(0) = 0$. In the present case, only the definition given in Eq. (2.8) is acceptable for $\bar{A}(0) = 0$. Also by definition, $F_{1\omega}(0) = 0$. Thus, Eq. (B.3) reduces to

$$BI(1) = (2/3u_0) \int_0^\xi [de_\omega(\xi')/d\xi'] \bar{A} \left[\frac{3}{2} u_0 (\xi - \xi') \right] d\xi' \quad (B.4)$$

The second integral in Eq. (3.21) is written as

$$BI(2) = (2/3u_0) \int_0^\xi [de_\omega(\xi')] \bar{A} \left[\frac{3}{2} u_0 (\xi' - \xi) \right] d\xi' \quad (B.5)$$

Let,

$$m = F_{2\omega}; \quad dn = \bar{A}' \left[\frac{3}{2} u_0 (\xi' - \xi) \right] d\xi'$$

Then,

$$dm = (dF_{2\omega}/d\xi')d\xi' = (de_{\omega}/d\xi')d\xi'$$

Now, to get n, let

$$u = \frac{3}{2} u_0 (\xi' - \xi), \quad du/d\xi' = 3u_0/2$$

and

$$\begin{aligned} dn &= [d\bar{A}(u)/du]d\xi' = [d\bar{A}(u)/d\xi' \times d\xi'/du]d\xi' \\ &= (2/3u_0)[d\bar{A}(u)/d\xi'] d\xi' = (2/3u_0) d\bar{A}(u) \end{aligned}$$

Thus,

$$n = (2/3u_0)\bar{A}(u) = (2/3u_0)\bar{A}\left[\frac{3}{2} u_0 (\xi' - \xi)\right]$$

Consequently, Eq. (B.5) is expressed as

$$\begin{aligned} BI(2) &= F_{2\omega}(1) \left\{ (2/3u_0) \bar{A} \left[\frac{3}{2} u_0 (1 - \xi) \right] \right\} - \left\{ F_{2\omega}(\xi) \left[(2/3u_0) \bar{A}(0) \right] \right\} \\ &\quad - (2/3u_0) \int_{\xi}^1 [de_{\omega}(\xi')/d\xi'] \bar{A} \left[\frac{3}{2} u_0 (\xi' - \xi) \right] d\xi' \end{aligned} \quad (B.6)$$

Since, by definition, $F_{2\omega}(1) = 0$ and $A(0) = 0$, Eq. (B.6) reduces to

$$BI(2) = - (2/3u_0) \int_{\xi}^1 [de_{\omega}(\xi')/d\xi'] \bar{A} \left[\frac{3}{2} u_0 (\xi' - \xi) \right] d\xi' \quad (B.7)$$

By use of Eqs. (B.4) and (B.7), the integrals in Eq. (3.21) can be expressed in alternate forms and this results in Eq. (3.23).

Consider now the integrals in Eq. (3.22); the first integral is written as

$$BI(3) = \int_0^{\xi} F_{1\omega}(\xi') \bar{A}'' \left[\frac{3}{2} u_0 (\xi - \xi') \right] d\xi' \quad (B.8)$$

For integration by parts, let

$$m = F_{1\omega}(\xi'); \quad dn = \bar{A}'' \left[\frac{3}{2} u_0 (\xi - \xi') \right] d\xi'$$

Then

$$dm = (dF_{1\omega}/d\xi') d\xi' = (de_{\omega}/d\xi') d\xi'$$

As before, to obtain n, let

$$u = \frac{3}{2} u_0 (\xi - \xi'), \quad du/d\xi' = - 3u_0/2$$

and, therefore,

$$\begin{aligned}
dn &= \frac{d}{du} [d\bar{A}(u)/du] d\xi' \\
&= \left\{ \frac{d}{d\xi'} \frac{d\xi'}{du} [d\bar{A}(u)/du] d\xi' \right. \\
&= d\{(-2/3 u_0)[d\bar{A}(u)/du]\}
\end{aligned}$$

Thus,

$$n = (-2/3 u_0) \bar{A}'(u) = (-2/3 u_0) \bar{A} \left[\frac{3}{2} u_0 (\xi - \xi') \right]$$

and Eq. (B.8) can be written as

$$\begin{aligned}
BI(3) &= \left\{ F_{1\omega} (-2/3 u_0) \bar{A}' \left[\frac{3}{2} u_0 (\xi - \xi') \right] \right\}_0^\xi \\
&- \int_0^\xi \left\{ (-2/3 u_0) \bar{A}' \left[\frac{3}{2} u_0 (\xi - \xi') \right] \right\} [dF_{1\omega}(\xi')/d\xi'] d\xi'. \quad (B.9)
\end{aligned}$$

Since $F_{1\omega}(0) = 0$ and $\bar{A}'(0) = 1$, Eq.(B.9) reduces to

$$BI(3) = (2/3 u_0) \left\{ -F_{1\omega}(\xi) + \int_0^\xi [de_\omega(\xi')/d\xi'] \bar{A}' \left[\frac{3}{2} u_0 (\xi - \xi') \right] d\xi' \right\} \quad (B.10)$$

Similarly, for the second integral in Eq. (3.22), one can find

$$\begin{aligned}
BI(4) &= \int_\xi^1 F_{2\omega}(\xi') \bar{A}'' \left[\frac{3}{2} u_0 (\xi' - \xi) \right] d\xi' \\
&= (-2/3 u_0) \left\{ F_2(\xi) + \int_\xi^1 [de_\omega(\xi')/d\xi'] \bar{A}' \left[\frac{3}{2} u_0 (\xi' - \xi) \right] d\xi' \right\} \quad (B.11)
\end{aligned}$$

A substitution of Eqs. (B.10) and (B.11) in Eq. (3.22) results in Eq. (3.24). Also, a differentiation of Eq. (3.23) with respect to ξ , by using the Leibnitz formula, gives

$$\begin{aligned}
 \frac{dq_r(\xi)}{d\xi} &= \sum_{i=1}^n A_{oi} \left\{ \frac{de_{\omega i}(\xi)}{d\xi} \bar{A}(0) - 0 \right. \\
 &+ \frac{3u_{oi}}{2} \int_0^{\xi} \left\{ \frac{de_{\omega i}(\xi')}{d\xi'} \bar{A}_i' \left[\frac{3}{2} u_{oi}(\xi - \xi') \right] d\xi' \right. \\
 &+ 0 - \frac{de_{\omega i}(\xi)}{d\xi} \bar{A}_i(0) \\
 &\left. \left. - \frac{3u_{oi}}{2} \int_{\xi}^1 \frac{de_{\omega i}(\xi')}{d\xi'} \bar{A}_i' \left[\frac{3}{2} u_{oi}(\xi') \right] d\xi' \right\} \right. \quad (B.12)
 \end{aligned}$$

Since $\bar{A}(0) = 0$, Eq. (B.12) reduces to Eq. (3.24).

APPENDIX C

DEFINITION AND EVALUATION OF INTEGRAL FUNCTIONS

For the convenience and use in the computational procedure, the following definitions are employed in expressing the relations for the integral functions:

$$\begin{aligned} b_i &= 3u_{oi}/2 \\ c_i &= 1/b_i = 2/3u_{oi} \\ C(\xi) &= 1/(\xi - \xi^2) \\ \Gamma &= 1/(\sigma T_1^3) \end{aligned}$$

Various integrals are defined and simplified as follows:

$$\begin{aligned} G_1(\xi) &= C(\xi) (3N_1(\xi - \xi^2) + \\ &+ \frac{9}{4} \sum_{i=1}^n M_{1i} u_{oi}^2 \{ \int_0^\xi (\xi' - \xi'^2) \bar{A}_i'' [b_i (\xi - \xi')] d\xi' \\ &+ \int_\xi^1 (\xi' - \xi'^2) \bar{A}_i'' [b_i (\xi' - \xi)] d\xi' \}) \end{aligned} \quad (C.1a)$$

$$\begin{aligned} &= C(\xi) (3N_1(\xi - \xi^2) + \\ &+ \frac{3}{2} \sum_{i=1}^n M_{1i} u_{oi} \{ \int_0^{b_i \xi} [\xi - c_i u - \xi^2 + 2\xi c_i u - (c_i u)^2] \bar{A}''(u) du \\ &+ \int_0^{b_i(1-\xi)} [\xi + c_i u - \xi^2 - 2\xi c_i u - (c_i u)^2] \bar{A}''(u) du \}) \end{aligned} \quad (C.1b)$$

$$\begin{aligned}
G_2(\xi) &= C(\xi) \left(3N_1 \left(\frac{1}{2} - \xi^2 \right) \right. \\
&\quad - \frac{9}{4} \sum_{i=1}^n (M_{1i} u_{0i}^2 \{ \int_0^\xi \xi'^2 A_i'' [b_i(\xi - \xi')] d\xi' \\
&\quad \left. + \int_\xi^1 (\xi'^2 - 1) \overline{A}_i'' [b_i(\xi' - \xi)] d\xi' \} \right) \quad (C.2a)
\end{aligned}$$

$$\begin{aligned}
&= C(\xi) \left(3N_1 \left(\frac{1}{2} - \xi^2 \right) \right. \\
&\quad - \frac{3}{2} \sum_{i=1}^n M_{1i} u_{0i} \{ \int_0^{b_i \xi} [\xi^2 - 2\xi c_i u + (c_i u)^2] \overline{A}_i''(u) du \\
&\quad \left. + \int_0^{b_i(1-\xi)} [\xi^2 + 2\xi c_i u + (c_i u)^2 - 1] \overline{A}_i''(u) du \} \right) \quad (C.2b)
\end{aligned}$$

$$\begin{aligned}
G_3(\xi) &= C(\xi) \left(\frac{3}{2} \sum_{i=1}^n N_{1i} \{ \int_0^\xi (1 - 2\xi') \overline{A}_i' [b_i(\xi - \xi')] d\xi' \right. \\
&\quad \left. - \int_\xi^1 (1 - 2\xi') \overline{A}_i' [b_i(\xi' - \xi)] d\xi' \} \right) \quad (C.3a)
\end{aligned}$$

$$\begin{aligned}
&= C(\xi) \left(\sum_{i=1}^n M_{1i} \{ \int_0^{b_i \xi} [1 - 2\xi + 2 c_i u] \overline{A}_i'(u) du \right. \\
&\quad \left. - \int_0^{b_i(1-\xi)} [1 - 2\xi - c_i u] \overline{A}_i'(u) du \} \right) \quad (C.3b)
\end{aligned}$$

$$\begin{aligned}
G_4(\xi) &= C(\xi) \left(-3 \sum_{i=1}^n N_{1i} \{ \int_0^\xi \xi' \overline{A}_i' [b_i(\xi - \xi')] d\xi' \right. \\
&\quad \left. - \int_\xi^1 \xi' \overline{A}_i' [b_i(\xi' - \xi)] d\xi' \} \right) \quad (C.4a)
\end{aligned}$$

$$\begin{aligned}
&= C(\xi) \left(-2 \sum_{i=1}^n M_{1i} \left\{ \int_0^{b_i \xi} [\xi - c_i u] \bar{A}_i'(u) du \right. \right. \\
&\quad \left. \left. + \int_{\xi}^{b_i(1-\xi)} [\xi + c_i u] \bar{A}_i'(u) du \right\} \right) \quad (C4.b)
\end{aligned}$$

$$\begin{aligned}
G_5(\xi) &= (3\Gamma/8) \sum_{i=1}^n u_{0i} H_{1i} \left\{ \int_0^{\xi} (\xi' - \xi')^2 \bar{A}_i' [b_i(\xi - \xi')] d\xi' \right. \\
&\quad \left. - \int_{\xi}^1 (\xi' - \xi')^2 \bar{A}_i' [b_i(\xi - \xi')] d\xi' \right\} \quad (C.5a)
\end{aligned}$$

$$\begin{aligned}
&= (\Gamma/4) \sum_{i=1}^n H_{1i} \left\{ \int_0^{b_i \xi} [\xi - c_i u - \xi^2 + 2\xi c_i u - (c_i u)^2] \bar{A}_i'(u) du \right. \\
&\quad \left. - \int_0^{b_i(1-\xi)} [\xi + c_i u - \xi^2 - 2\xi c_i u - (c_i u)^2] \bar{A}_i'(u) du \right\} \quad (C.5b)
\end{aligned}$$

$$\begin{aligned}
G_6(\xi) &= (3\Gamma/8) \sum_{i=1}^n u_{0i} H_{1i} \left\{ \int_0^{\xi} \xi'^2 \bar{A}_i' [b_i(\xi - \xi')] d\xi' \right. \\
&\quad \left. + \int_{\xi}^1 (1 - \xi')^2 \bar{A}_i' [b_i(\xi' - \xi)] d\xi' \right\} \quad (C.6a)
\end{aligned}$$

$$\begin{aligned}
&= (\Gamma/4) \sum_{i=1}^n H_{1i} \left\{ \int_0^{b_i \xi} [\xi^2 - 2\xi c_i u + (c_i u)^2] \bar{A}_i'(u) du \right. \\
&\quad \left. + \int_0^{b_i(1-\xi)} [1 - \xi^2 - 2\xi c_i u - (c_i u)^2] \bar{A}_i'(u) du \right\} \quad (C.6b)
\end{aligned}$$

$$G_7(\xi) = (r/4) \sum_{i=1}^n H_{1i} \left\{ \int_0^\xi (1-2\xi') \bar{A}_i [b_i(\xi-\xi')] d\xi' \right. \\ \left. - \int_\xi^1 (1-2\xi') \bar{A}_i [b_i(\xi'-\xi)] d\xi' \right\} \quad (C.7a)$$

$$= (r/6) \sum_{i=1}^n (H_{1i}/u_{0i}) \left\{ \int_0^{b_i\xi} (1-2\xi + 2c_i u) \bar{A}_i(u) du \right. \\ \left. - \int_0^{b_i(1-\xi)} (1-2\xi-2c_i u) \bar{A}_i(u) du \right\} \quad (C.7b)$$

$$G_8(\xi) = (r/4) \sum_{i=1}^n H_{1i} \left\{ \int_0^\xi (2\xi') \bar{A}_i [b_i(\xi-\xi')] d\xi' \right. \\ \left. - \int_\xi^1 (2\xi') \bar{A}_i [b_i(\xi'-\xi)] d\xi' \right\} \quad (C.8a)$$

$$= (r/6) \sum_{i=1}^n (H_{1i}/u_{0i}) \left\{ \int_0^{b_i\xi} (2\xi - 2c_i u) \bar{A}_i(u) du \right. \\ \left. - \int_0^{b_i(1-\xi)} (2\xi + 2c_i u) \bar{A}_i(u) du \right\} \quad (C.8b)$$

$$G_9 = (3r/8) \sum_{i=1}^n u_{0i} H_{1i} \left\{ \int_0^1 (\xi' - \xi'^2) \bar{A}_i(b_i \xi') d\xi' \right\} \quad (C.9a)$$

$$= (r/4) \sum_{i=1}^n H_{1i} \left\{ \int_0^{b_i} [c_i u - (c_i u)^2] \bar{A}_i(u) du \right\} \quad (C.9b)$$

$$G_{10} = (3\Gamma/8) \sum_{i=1}^n u_{0i} H_{1i} \left\{ \int_0^1 (1-\xi') \bar{A}_i (b_i \xi') d\xi' \right\} \quad (C.10a)$$

$$= (\Gamma/4) \sum_{i=1}^n H_{1i} \left\{ \int_0^{b_i} (1-c_i u) \bar{A}_i (u) du \right\} \quad (C.10b)$$

$$G_{11} = (\Gamma/4) \sum_{i=1}^n H_{1i} \left\{ \int_0^1 (1-2\xi') \bar{A}_i (b_i \xi') d\xi' \right\} \quad (C.11a)$$

$$= (\Gamma/6) \sum_{i=1}^n (H_{1i} \left\{ \int_0^{b_i} (1-2c_i u) \bar{A}_i (u) du \right\}) \quad (C.11b)$$

$$G_{12} = (\Gamma/4) \sum_{i=1}^n (H_{1i}/u_{0i}) \left\{ \int_0^1 (2\xi') \bar{A}_i (b_i \xi') d\xi' \right\} \quad (C.12a)$$

$$= (\Gamma/6) \sum_{i=1}^n (H_{1i}/u_{0i}) \left\{ \int_0^{b_i} (2c_i u) \bar{A}_i (u) du \right\} \quad (C.12b)$$

$$G_{13} (\xi) = C(\xi) \left\{ 3N_1 (\xi - \xi^2) - M_1 \left[\int_0^\xi (\xi' - \xi'^2) \frac{d\xi'}{(\xi - \xi')^2} \right. \right. \\ \left. \left. + \int_\xi^1 (\xi' - \xi'^2) \frac{d\xi'}{(\xi' - \xi)^2} \right] \right\} \quad (\text{improper}) \quad (C.13)$$

$$G_{14} (\xi) = C(\xi) \left\{ 3N_1 \left(\frac{1}{2} - \xi^2 \right) + M_1 \left[\int_0^\xi \xi'^2 \frac{d\xi'}{(\xi - \xi')^2} \right. \right. \\ \left. \left. + \int_\xi^1 (\xi'^2 - 1) \frac{d\xi'}{(\xi' - \xi)^2} \right] \right\} \quad (\text{improper}) \quad (C.14)$$

$$G_{15}(\xi) = M_1 C(\xi) \int_0^1 (1-2\xi') \frac{d\xi'}{(\xi-\xi')} \quad (C.15a)$$

$$= M_1 C(\xi) \{2 + (2\xi-1) \ln [(\xi-1)/\xi]\} \quad (C.15b)$$

$$G_{16}(\xi) = -2M_1 C(\xi) \int_0^1 \xi' \frac{d\xi'}{(\xi-\xi')} \quad (C.16a)$$

$$= 2 M_1 C(\xi) \{1 + \xi \ln [(\xi-1)/\xi]\} \quad (C.16b)$$

$$G_{17}(\xi) = (\Gamma H_1/4) \int_0^1 (\xi' - \xi'^2) \frac{d\xi'}{(\xi-\xi')} \quad (C.17a)$$

$$= (\Gamma H_1/4) \left\{ -\frac{1}{2} + \xi - (\xi - \xi^2) \ln [(\xi-1)/\xi] \right\} \quad (C.17b)$$

$$G_{18}(\xi) = (\Gamma H_1/4) \left[\int_0^1 \xi'^2 \frac{d\xi'}{(\xi-\xi')} - \int_{\xi}^1 \frac{d\xi'}{(\xi-\xi')} \right] \text{ (improper)} \quad (C.18)$$

$$G_{19}(\xi) = (\Gamma/4) \sum_{i=1}^n H_{1i} \left\{ \int_0^{\xi} (1-2\xi') \ln [b_i(\xi-\xi')] d\xi' \right.$$

$$\left. + \int_{\xi}^1 (1-2\xi') \ln [b_i(\xi'-\xi)] d\xi' \right\} \quad (C.19a)$$

$$= (\Gamma H_1/4) \left\{ -\frac{1}{2} + \xi + (\xi^2 - \xi) \ln [(1-\xi)/\xi] \right\} \quad (C.19b)$$

$$G_{20}(\xi) = (\Gamma/4) \sum_{i=1}^n H_{1i} \left\{ \int_0^{\xi} 2\xi' \ln [b_i(\xi-\xi')] d\xi' \right.$$

$$+ \int_{\xi}^1 2\xi' \ln [b_i (\xi' - \xi)] d\xi' \quad (C.20a)$$

$$= (r/4) (H_1 \{ \xi^2 \ln [\xi/(1-\xi)] - \xi - \frac{1}{2} \}$$

$$+ \sum_{i=1}^n H_{1i} \ln [b_i (1-\xi)]) \quad (C.20b)$$

$$G_{21} = (rH_1/4) \int_0^1 (\xi' - \xi'^2) d\xi'/\xi' = (rH_1/8) \quad (C.21)$$

$$G_{22} = (rH_1/4) \int_0^1 (1 - \xi'^2) d\xi'/\xi' \quad (C.22a)$$

$$= - (rH_1/4) \left[\frac{1}{2} + \ln(0) \right] \quad (\text{improper}) \quad (C.22b)$$

$$G_{23} = (r/4) \sum_{i=1}^n H_{1i} \int_0^1 (1 - 2\xi') \ln (b_i \xi') d\xi' \quad (C.23a)$$

$$= - (r H_1/8) = - H_1/(8\sigma T_1^3) \quad (C.23b)$$

$$G_{24} = (r/4) \sum_{i=1}^n H_{1i} \int_0^1 2\xi' \ln (b_i \xi') d\xi' \quad (C.24a)$$

$$= (r/4) \left[- (H_1/2) + \sum_{i=1}^n H_{1i} \ln (b_i) \right] \quad (C.24b)$$

APPENDIX D

INFORMATION FOR NUMERICAL PROCEDURE

1. Data:

$$T = 300, 500, 1000, 2000, 5000, \text{ etc. } \sim K$$

$$P = 0.1, 1, 2, 5, 10, \text{ etc. } \sim \text{atm}$$

$$L = 1, 2, 5, 10, 20, \text{ etc. } \sim \text{cm}$$

$$\xi \rightarrow x = 0.0 \rightarrow 1.0; x = U(I)$$

2. Thermal conductivity of the gas:

$$K_f = K_f (T, P) \quad (D.1)$$

3. The Planck function and its derivative:

$$e_{b\omega}(T) = \frac{C_1}{\exp(C_2/T) - 1} \equiv PF \quad (D.2)$$

$$\frac{de_{b\omega}}{dT} = \frac{C_1 C_2 \exp(C_2/T)}{T^2 [\exp(C_2/T) - 1]^2} \equiv PFD \quad (D.3)$$

where

$$C_1 = (2\pi hc^2)\omega^3$$

$$C_2 = (hc/k) \omega$$

$$C_1 C_2 = (2\pi h^2 c^3/k)\omega^4$$

and $\omega = \omega_c = \omega_0$ = wave number center of the i th band.

By defining

$$\text{TEXP} = \exp(C_2/T)$$

Eq. (D.3) is expressed as

$$\text{PFD} = (C_1 C_2)(\text{TEXP})/[(T^2)(\text{TEXP} - 1)^2] \quad (\text{D.4})$$

All values in Eqs. (D.2) - (D.4) should be evaluated for each band.

4. Information or relation for A_0 for each band, $A_0 = f(T)$.
5. Information on C_0^2 for each band, $C_0^2 = f(T)$.
6. Information on u_0 for each band, $u_0 = C_0^2 PL$ (nondimensional).
7. Information on $B^2(T)$ for each band (nondimensional).
8. Equivalent or effective pressure relation for each band.

$$P_{ei} = [(P/P_0) + (P_i/P_0)(b_i - 1)]^n \quad (\text{nondimensional}) \quad (\text{D.5})$$

In Eq. (D.5), $P_0 = 1$ atm and, therefore, P_i and P must be in the units of atm. Note that P_i is the partial pressure of the i th species in a gases mixture and P is the total pressure; b_i (the self broadening coefficient) and n are different for different bands. For a single component system, Eq. (D.5) is usually expressed as

$$P_e = (b P/P_0)^n \quad (\text{nondimensional}) \quad (\text{D.6})$$

9. Line structure parameter for each band:

$$\beta = B^2 P_e \equiv \text{BETA} \quad (\text{D.7})$$

10. Correlation for each band (for Tien and Lowder's correlation):

$$f(\beta) = 2.94 [1 - \exp(-2.60\beta)] \equiv F \quad (\text{D.8})$$

11. Band absorptance correlation for each band (Tien and Lowder's correlation):

$$\bar{A}(u, \beta) = \ln \left[uF \left(\frac{u + 2}{u + 2F} \right) + 1 \right] \equiv AU \quad (\text{D.9})$$

12. The derivative of the band absorptance correlation for each band (Tien and Lowder's correlation)

$$\bar{A}'(u, \beta) = [F(u^2 + 4uF + 4F)]/\text{DEN} \equiv AUD \quad (\text{D.10})$$

where

$$\text{DEN} = [F(u^2 + 2u + 2) + u](u + 2F)$$

These basic relations are used in the governing equations of Section 6 to obtain numerical solutions for specific gaseous systems. The spectroscopic and correlation quantities needed for these calculations are available in [22,24].

APPENDIX E

EVALUATION OF CONSTANTS FOR STEADY LAMINAR FLOWS

To determine the constants in Eq. (7.47), Eq. (7.49) is evaluated at $\xi = 0$ and $\xi = 1/4$. To avoid excessive writing, the following notations are used (some of which are also used in Appendix C)

$$\xi' = \eta, \quad d\xi' = d\eta$$

$$b_i = 3u_{0i}/2; \quad c_i = 1/b_i = 2/3u_{0i}$$

For $\xi = 0$, Eq. (7.49) reduces to

$$\begin{aligned} a_1 + 1 + \frac{3}{2} (L/k) \sum_{i=1}^n H_{1i} u_{0i} \left\{ \int_0^1 [a_1 (\eta - 2\eta^3 + \eta^4) \right. \\ \left. + a_2 (\eta^2 - 2\eta^3 + \eta^4)] \bar{A}_i'(b_i \eta) d\eta \right\} = 0 \end{aligned} \quad (E.1)$$

By defining $u = b_i \eta$, Eq. (E.1) is expressed in an alternate form as

$$\begin{aligned} a_1 \left\{ 1 + (L/k) \sum_{i=1}^n H_{1i} \int_0^{b_i} [c_i u - 2(c_i u)^3 + (c_i u)^4] \bar{A}_i'(u) du \right\} \\ + a_2 \left\{ (L/k) \sum_{i=1}^n H_{1i} \int_0^{b_i} [(c_i u)^2 - 2(c_i u)^3 + (c_i u)^4] \bar{A}_i'(u) du \right\} = -1 \end{aligned} \quad (E.2)$$

Now, by defining the following integral functions,

$$R_{1i} = \int_0^{b_i} u_i \bar{A}_i'(u_i) du_i \quad (\text{E.3a})$$

$$R_{2i} = \int_0^{b_i} u_i^2 \bar{A}_i'(u_i) du_i \quad (\text{E.3b})$$

$$R_{3i} = \int_0^{b_i} u_i^3 \bar{A}_i'(u_i) du_i \quad (\text{E.3c})$$

$$R_{4i} = \int_0^{b_i} u_i^4 \bar{A}_i'(u_i) du_i \quad (\text{E.3d})$$

Eq. (E.2) is expressed as

$$a_1 \alpha_1 + a_2 \alpha_2 = -1 \quad (\text{E.4a})$$

where

$$\alpha_1 = 1 + (L/k) \sum_{i=1}^n H_{1i} (c_i R_{1i} - 2 c_i^3 R_{3i} + c_i^4 R_{4i}) \quad (\text{E.4b})$$

$$\alpha_2 = (L/k) \sum_{i=1}^n H_{1i} (c_i^2 R_{2i} - 2 c_i^3 R_{3i} + c_i^4 R_{4i}) \quad (\text{E.4c})$$

For $\xi = 1/4$, Eq. (7.49) reduces to

$$\begin{aligned} & \frac{11}{16} a_1 + \frac{3}{16} a_2 + \frac{11}{16} \\ &= \frac{3}{2} (L/k) \sum_{i=1}^n H_{1i} u_{0i} \left\{ \int_0^{1/4} \theta(n) \bar{A}_i' \left[b_i \left(\frac{1}{4} - n \right) \right] dn \right. \\ & \quad \left. - \int_{1/4}^1 \theta(n) \bar{A}_i' \left[b_i \left(n - \frac{1}{4} \right) \right] dn \right\} \end{aligned} \quad (\text{E.5})$$

By defining $u = b_i (\frac{1}{4} - \eta)$ for the first integral and $u = b_i (\eta - \frac{1}{4})$ for the second integral, Eq. (E.5) is written as

$$\begin{aligned} \frac{11}{16} + \frac{11}{16} a_1 + \frac{3}{16} a_2 = (L/k) \sum_{i=1}^n H_{1i} \{ \int_0^{b_i/4} \theta(\frac{1}{4} - c_i u) \bar{A}_i'(u) du \\ - \int_0^{3b_i/4} \theta(\frac{1}{4} + c_i u) \bar{A}_i'(u) du \end{aligned} \quad (E.6)$$

By denoting $d = c_i u$, the following relations are obtained from Eq. (7.47):

$$\begin{aligned} \theta(\frac{1}{4} - d) = a_1 (\frac{57}{256} - \frac{11}{16} d - \frac{9}{8} d^2 + d^3 + d^4) \\ + a_2 (\frac{9}{256} - \frac{13}{16} d - \frac{1}{8} d^2 - d^3 + d^4) \end{aligned} \quad (E.7a)$$

and

$$\begin{aligned} \theta(\frac{1}{4} + d) = a_1 (\frac{57}{256} + \frac{11}{16} d - \frac{9}{8} d^2 - d^3 + d^4) \\ + a_2 (\frac{9}{256} + \frac{3}{16} d - \frac{1}{8} d^2 - d^3 + d^4) \end{aligned} \quad (E.7b)$$

A combination of Eqs. (E.6) and (E.7) results in

$$\begin{aligned} a_1 (\frac{L}{K} \sum_{i=1}^n H_{1i} \{ \frac{57}{256} [\int_0^{b_i/4} \bar{A}_i'(u) du - \int_0^{3b_i/4} \bar{A}_i'(u) du] \\ - \frac{11}{16} c_i [\int_0^{b_i/4} u \bar{A}_i'(u) du + \int_0^{3b_i/4} u \bar{A}_i'(u) du] \end{aligned}$$

$$\begin{aligned}
& - \frac{9}{8} c_i^2 \left[\int_0^{b_i/4} u^2 \overline{A}_i'(u) du - \int_0^{3b_i/4} u^2 \overline{A}_i'(u) du \right] \\
& + c_i^3 \left[\int_0^{b_i/4} u^3 \overline{A}_i'(u) du + \int_0^{3b_i/4} u^3 \overline{A}_i'(u) du \right] \\
& + c_i^4 \left[\int_0^{b_i/4} u^4 \overline{A}_i'(u) du - \int_0^{3b_i/4} u^4 \overline{A}_i'(u) du \right] \} - \frac{11}{16} \\
& + a_2 \left(\frac{L}{K} \sum_{i=1}^n H_{1i} \left\{ \frac{9}{256} \left[\int_0^{b_i/4} \overline{A}_i'(u) du - \int_0^{3b_i/4} \overline{A}_i'(u) du \right] \right. \right. \\
& \quad - \frac{3}{16} c_i \left[\int_0^{b_i/4} u \overline{A}_i'(u) du + \int_0^{3b_i/4} u \overline{A}_i'(u) du \right] \\
& \quad - \frac{1}{8} c_i^2 \left[\int_0^{b_i/4} u^2 \overline{A}_i'(u) du - \int_0^{3b_i/4} u^2 \overline{A}_i'(u) du \right] \\
& \quad + c_i^3 \left[\int_0^{b_i/4} u^3 \overline{A}_i'(u) du + \int_0^{3b_i/4} u^3 \overline{A}_i'(u) du \right] \\
& \quad \left. + c_i^4 \left[\int_0^{b_i/4} u^4 \overline{A}_i'(u) du - \int_0^{3b_i/4} u^4 \overline{A}_i'(u) du \right] \right\} - \frac{3}{16} \Big) \\
& = 11/16
\end{aligned} \tag{E.8}$$

By noting that for any continuous function $F(x)$

$$\int_0^{3/4} F(x) dx = \int_0^{1/4} F(x) dx + \int_{1/4}^{3/4} F(x) dx$$

and defining

$$S_{1i} = \int_{b_i/4}^{3b_i/4} \bar{A}_i'(u_i) du_i \quad (\text{E.9a})$$

$$S_{2i} = \int_0^{b_i/4} u_i \bar{A}_i'(u_i) du_i \quad (\text{E.9b})$$

$$S_{3i} = \int_0^{3b_i/4} u_i \bar{A}_i'(u_i) du_i \quad (\text{E.9c})$$

$$S_{4i} = \int_{b_i/4}^{3b_i/4} u_i^2 \bar{A}_i'(u_i) du_i \quad (\text{E.9d})$$

$$S_{5i} = \int_0^{b_i/4} u_i^3 \bar{A}_i'(u_i) du_i \quad (\text{E.9e})$$

$$S_{6i} = \int_0^{3b_i/4} u_i^3 \bar{A}_i'(u_i) du_i \quad (\text{E.9f})$$

$$S_{7i} = \int_{b_i/4}^{3b_i/4} u_i^4 \bar{A}_i'(u_i) du_i \quad (\text{E.9g})$$

Eq. (E.8) can be written as

$$a_1 \alpha_3 + a_2 \alpha_4 = -11/16 \quad (\text{E.10a})$$

where

$$\alpha_3 = (L/k) \sum_{i=1}^n H_{1i} \left[(57/256) S_{1i} + (11/16) c_i (S_{2i} + S_{3i}) \right. \\ \left. - (9/8) c_i^2 S_{4i} - c_i^3 (S_{5i} + S_{6i}) + c_i^4 S_{7i} \right] + 11/16 \quad (E.10b)$$

$$\alpha_4 = (L/k) \sum_{i=1}^n H_{1i} \left[(9/256) S_{1i} + (3/16) c_i (S_{2i} + S_{3i}) \right. \\ \left. - (1/8) c_i^2 S_{4i} - c_i^3 (S_{5i} + S_{6i}) + c_i^4 S_{7i} \right] + 3/16 \quad (E.10c)$$

By solving Eqs. (E.4a) and (E.10a) simultaneously, there is obtained the results for constants a_1 and a_2 as

$$a_1 = (11 \alpha_2 - 16 \alpha_4) / \text{DEN} \quad (E.11a)$$

$$a_2 = (16 \alpha_3 - 11 \alpha_1) / \text{DEN} \quad (E.11b)$$

where

$$\text{DEN} = 16 (\alpha_1 \alpha_4 - \alpha_2 \alpha_3) \quad (E.11c)$$

The governing equation for the large path length limit is Eq. (7.44a) for which the solution is also given by Eq. (7.47). For the large path length limit, Eq. (7.47) is expressed in the form of Eq. (7.52) which is represented here as

$$\theta(\xi) = b_1 (\xi - 2\xi^3 + \xi^4) + b_2 (\xi^2 - 2\xi^3 + \xi^4) \quad (E.12)$$

Thus,

$$\theta'(\xi) = b_1 (1 - 6\xi^2 + 4\xi^3) + b_2 (2\xi - 6\xi^2 + 4\xi^3) \quad (E.13)$$

C.2

A substitution of Eqs. (E.12) and (E.13) in Eq. (7.44a) gives

$$\begin{aligned} b_1 (1 - 6\xi^2 + 4\xi^3) + b_2 (2\xi - 6\xi^2 + 4\xi^3) - 2 (3\xi^2 - 2\xi^3) + 1 \\ = M \int_0^1 \theta(\xi') d\xi' / (\xi - \xi') \end{aligned} \quad (\text{E.14})$$

For $\xi = 0$, Eq. (E.14) reduces to

$$b_1 + 1 = - M \int_0^1 [\theta(\xi') / \xi'] d\xi' \quad (\text{E.15})$$

Upon substituting for $\theta(\xi')$ from Eq. (E.12) into Eq. (E.15), the integrals can be evaluated in closed form and there is obtained

$$b_1 \beta_1 + b_2 \beta_2 = -1 \quad (\text{E.16a})$$

where

$$\beta_1 = 1 + (7/12) M; \quad \beta_2 = (1/12) M \quad (\text{E.16b})$$

For $\xi = 1/4$, Eq. (E.14) reduces to

$$\frac{11}{16} b_1 + \frac{3}{16} b_2 + \frac{11}{16} = M \int_0^1 \theta(\xi') d\xi' / \left(\frac{1}{4} - \xi'\right) \quad (\text{E.17})$$

By substituting for $\theta(\xi')$ in this equation, another relation between b_1 and b_2 can be obtained in terms of β_3 and β_4 . But, this appears to involve the evaluation of a few improper integrals. Thus, this approach is abandoned in

favor of an alternate procedure discussed below.

The solution in the large path length limit can be obtained from the general solution by evaluating the integrals R_i and S_i in the limit of large path length. Since in this limit $\bar{A}'(u) = 1/u$, the integrals in Eqs. (E.3) and (E.9) are evaluated to obtain

$$R_{1i} = b_i; R_{2i} = b_i^2/2; R_{3i} = b_i^3/3; R_{4i} = b_i^4/4 \quad (\text{E.18a})$$

$$\begin{aligned} S_{1i} &= \ln(3); S_{2i} = b_i/4; S_{3i} = 3b_i/4; S_{4i} = b_i^2/4; \\ S_{5i} &= b_i^3/192; S_{6i} = 9b_i^3/64; S_{7i} = 5b_i^4/64 \end{aligned} \quad (\text{E.18b})$$

From Eqs. (E.4) and (E.18a), there is obtained for the large path length limit

$$b_1 \beta_1 + b_2 \beta_2 = -1 \quad (\text{E.19a})$$

$$\beta_1 = 1 + (7/12) M; \beta_2 = (1/12) M \quad (\text{E.19b})$$

which is the same result as given by Eq. (E.16). From Eqs. (E.10) and (E.18b), one obtains in the large path length limit

$$b_1 \beta_3 + b_2 \beta_4 = -11/16 \quad (\text{E.20a})$$

where

$$\begin{aligned} \beta_3 &\approx 11/16 + M [(57/256) \ln(3) + 65/192] \\ &\approx 11/16 + 0.583154559 M \end{aligned} \quad (\text{E.20b})$$

$$\begin{aligned}\beta_4 &= 3/16 + M [(9/256) \ln(3) + 17/192] \\ &= 3/16 + 0.127164755 M\end{aligned}\quad (\text{E.20c})$$

The solution of Eqs. (E.19a) and (E.20a) yields

$$b_1 = (11\beta_2 - 16\beta_4)/\text{BOTTOM} \quad (\text{E.21a})$$

$$b_2 = (16\beta_3 - 11\beta_1)/\text{BOTTOM} \quad (\text{E.21b})$$

where

$$\text{BOTTOM} = 16 (\beta_1\beta_4 - \beta_2\beta_3) \quad (\text{E.21c})$$

With b_1 and b_2 known, the solution for the temperature distribution is obtained from Eq. (E.12).

APPENDIX F

INTEGRAL FUNCTIONS FOR TRANSIENT LAMINAR FLOWS

For convenience and use in the computational procedure, the following definitions are employed in expressing the relations for the integral functions:

$$b_i = 3 u_{oi}/2; c_i = 1/b_i; C(\xi) = 1/(\xi - \xi^2)$$

$$x = \xi - 2\xi^3 + \xi^4; y = \xi^2 - 2\xi^3 + \xi^4; z = 12 (\xi - \xi^2)$$

$$a(\xi) = 1 - 6\xi^2 + 4\xi^3; b(\xi) = 2\xi - 6\xi^2 + 4\xi^3$$

Various integrals are defined as follows:

$$J_1(\xi) = 3 N + C(\xi) \left(2 + \frac{9}{4} (L/k) \sum_{i=1}^n H_{1i} u_{oi}^2 \left\{ \int_0^\xi (\xi' - \xi'^2) \overline{A}_i'' [b_i(\xi - \xi')] d\xi' \right. \right. \\ \left. \left. + \int_\xi^1 (\xi' - \xi'^2) \overline{A}_i'' [b_i(\xi' - \xi)] d\xi' \right\} \right) \quad (F.1)$$

$$J_2(\xi) = C(\xi) \left(2 + \frac{3}{2} (L/k) \sum_{i=1}^n H_{1i} u_{oi}^2 \left\{ \int_0^\xi (1 - 2\xi') \overline{A}_i' [b_i(\xi - \xi')] d\xi' \right. \right. \\ \left. \left. - \int_\xi^1 (1 - 2\xi') \overline{A}_i' [b_i(\xi' - \xi)] d\xi' \right\} \right) \quad (F.2)$$

$$J_3(\xi) = z + 3Nx + \frac{9}{4} (L/k) \sum_{i=1}^n H_{1i} u_{oi}^2 \left\{ \int_0^\xi x(\xi') \overline{A}_i'' [b_i(\xi - \xi')] d\xi' \right.$$

$$+ \int_{\xi}^1 x(\xi') \bar{A}_i'' [b_i(\xi' - \xi)] d\xi' \quad (F.3)$$

$$J_4(\xi) = (z-2) + 3Ny + \frac{9}{4} (L/k) \sum_{i=1}^n H_{1i} u_{0i}^2 \left\{ \int_0^{\xi} y(\xi') \bar{A}_i'' [b_i(\xi - \xi')] d\xi' \right. \\ \left. + \int_{\xi}^1 y(\xi') \bar{A}_i'' [b_i(\xi - \xi')] d\xi' \right\} \quad (F.4)$$

$$J_5(\xi) = z + \frac{3}{2} (L/k) \sum_{i=1}^n H_{1i} u_{0i} \left\{ \int_0^{\xi} a(\xi') \bar{A}_i' [b_i(\xi - \xi')] d\xi' \right. \\ \left. - \int_{\xi}^1 a(\xi') \bar{A}_i' [b_i(\xi' - \xi)] d\xi' \right\} \quad (F.5)$$

$$J_6(\xi) = (z - 2) + \frac{3}{2} (L/k) \sum_{i=1}^n H_{1i} u_{0i} \left\{ \int_0^{\xi} b(\xi') \bar{A}_i' [b_i(\xi - \xi')] d\xi' \right. \\ \left. - \int_{\xi}^1 b(\xi') \bar{A}_i' [b_i(\xi' - \xi)] d\xi' \right\} \quad (F.6)$$

$$J_7(\xi) = 3N + C(\xi) \left[2 + M \int_0^1 (1 - 2\xi') d\xi' / (\xi - \xi') \right] \\ = 3N + C(\xi) \left\{ 2 + M \left[2 + (2\xi - 1) \ln \left(\frac{\xi - 1}{\xi} \right) \right] \right\} \quad (F.7)$$

$$J_8(\xi) = J_7(\xi) - 3N \quad (F.8)$$